SINGULAR OSCILLATORY INTEGRALS IN EQUIVARIANT COHOMOLOGY. RESIDUE FORMULAE FOR BASIC DIFFERENTIAL FORMS ON GENERAL SYMPLECTIC MANIFOLDS

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Abstract. Let M be a symplectic manifold and G a connected, compact Lie group acting on M in a Hamiltonian way. In this paper, we study the equivariant cohomology of M represented by basic differential forms, and relate it to the cohomology of the Marsden-Weinstein reduced space via certain residue formulae using resolution of singularities and the stationary phase principle. In case that M is a compact, symplectic manifold or the co-tangent bundle of a G-manifold, similar residue formulae were derived by Jeffrey, Kirwan et al. [12, 11] for general equivariantly closed forms and by Ramacher [21] for basic differential forms, respectively.

Contents

1. Introduction

Consider a symplectic manifold (M, ω) carrying a Hamiltonian action of a connected, compact Lie group G with Lie algebra $\mathfrak{g}$. In case that M is compact and 0 a regular value of the momentum map $J : M \to \mathfrak{g}^*$, the cohomology of the Marsden-Weinstein reduced space $M_{\text{red}} := J^{-1}(0)/G$ was expressed by Jeffrey and Kirwan [12] in terms of the equivariant cohomology of M via certain residue formulae. Similar residue formulae were derived by them and their collaborators [11] for nonsingular, connected, complex, projective varieties M in case that 0 is not a regular value. In the latter case, analogous residue formulae were proved for basic differential forms in [21] if M is the co-tangent bundle of a G-manifold. In what follows, we shall extend these results to general, in particular non-compact, symplectic manifolds, and derive similar residue formulae by means of the stationary phase principle and resolution of singularities.

This paper is a continuation of the work initiated in [21], though the desingularization process implemented here is different and conceptually more natural, since it is carried out entirely within the symplectic category, and based on the Marle-Guillemin-Sternberg local normal form of the momentum

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map and certain discretized blow-ups. The crucial step in our approach of proving residue formulae consists in determining the asymptotic behavior of oscillatory integrals of the form

$$I_\eta(\mu) = \int_{\mathfrak{g}} \left[\int_M e^{i(J(p)-\eta)(X)/\mu} a(p,X)\, dM(p) \right] dX, \qquad \mu \to 0^+,$$

where $\eta \in \mathfrak{g}^*$, $a \in \mathrm{C}^\infty_\mathrm{c}(M \times \mathfrak{g})$ is an amplitude, dM the symplectic volume form on M, and dX denotes an Euclidean measure on $\mathfrak{g}$ given by an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$. If η is a regular value of the momentum map, the critical set of the phase function $(J(p)-\eta)(X)$ is clean, and an application of the stationary phase principle yields a complete asymptotic expansion. Nevertheless, serious difficulties arise when η is not a regular value. The reason is that in this case singular orbits occur. As a consequence, the critical set in question is no longer smooth, so that, a priori, the stationary phase principle can not be applied in this case. To overcome this problem in case where $\eta = 0$ is not a regular value we shall develop an iterative desingularization process, where in each step the Marle-Guillemin-Sternberg local normal form is employed in order to decompose the momentum map into a linear and a quadratic term, the first giving rise to very elementary oscillatory integrals of the form

$$\int_{\mathbb{R}^{2n}} e^{i\langle x,\xi\rangle/\nu} f(x,\xi)\, dx\, d\xi, \qquad f \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{2n}), \qquad \nu \to 0^+, \tag{1.1}$$

while the quadratic part is factorized via discretized blow-ups to yield a new phase function with a less singular critical set. After finitely many iterations, the quadratic term vanishes and the desingularization process comes to an end. By this we are able to compute the leading term in the asymptotic expansion of $I_0(\mu)$ together with a remainder estimate in case that 0 is not a regular value.

Together with the complete asymptotic expansion of $I_\eta(\mu)$ for regular values η this allows us to derive the following residue formula. Let $[\varrho] \in H^*_G(M)_c$ be an equivariant cohomology class of compact support represented by $\varrho(X) = \alpha + \beta(X)$, where α is a basic[1]differential form on M of compact support, and β is an equivariantly exact differential form of compact support. Fix a maximal torus $T \subset G$, and denote the corresponding root system by $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$. Assume that the reduced space M_red is connected, and denote its open, connected and dense stratum by $\mathrm{Reg}\, M_\mathrm{red} := (\Omega \cap M_{(H^\Omega_\mathrm{reg})})/G$, where $\Omega := J^{-1}(0)$, and $M_{(H^\Omega_\mathrm{reg})}$ is the stratum of M of orbit type (H^Ω_reg), H^Ω_reg being a certain closed subgroup of G.[2] We assume that H^Ω_reg is a finite group, which is equivalent to the assumption that 0 is a regular value of the restriction of J to $M_{(H^\Omega_\mathrm{reg})}$. Let ω_red be the unique symplectic form on $\mathrm{Reg}\, M_\mathrm{red}$ characterized by the condition $i^*\omega = \pi^*\omega_\mathrm{red}$, where $i : \Omega \cap M_{(H^\Omega_\mathrm{reg})} \to M$ is the inclusion and $\pi : (\Omega \cap M_{(H^\Omega_\mathrm{reg})}) \to \mathrm{Reg}\, M_\mathrm{red}$ is the canonical projection. Consider the so-called *Kirwan map*

$$\mathcal{K} : H^*_G(M)_c \xrightarrow{i^*} H^*_G(\Omega \cap M_{(H^\Omega_\mathrm{reg})}) \xrightarrow{(\pi^*)^{-1}} H^*(\mathrm{Reg}\, M_\mathrm{red}), \tag{1.2}$$

and note that the second map in this composition locally acts on a basic form $\eta \in \Omega^*_G(\Omega \cap M_{(H^\Omega_\mathrm{reg})})$ simply by

$$\big((\pi^*)^{-1}\eta\big)|_U = j^*(\eta|_{\pi^{-1}(U)}),$$

where $U \subset \mathrm{Reg}\, M_\mathrm{red}$ is a small open set and $j : U \to \pi^{-1}(U) \subset \Omega \cap M_{(H^\Omega_\mathrm{reg})}$ is an arbitrary smooth section of π on U. Then, by Theorem 9.2,

$$(-2\pi i)^d \int_{\mathrm{Reg}\, M_\mathrm{red}} e^{-i\omega_\mathrm{red}}\, \mathcal{K}(\alpha) = \frac{|H^\Omega_\mathrm{reg}|}{|W|\,\mathrm{vol}\, T} \mathrm{Res}\Big(\Phi^2 \sum_{F\in\mathcal{F}} u_F\Big), \tag{1.3}$$

where $d = \dim G$, $\mathcal{F}$ denotes the set of components of the fixed point set of the T-action on M, Φ denotes the product of the positive roots, W is the Weyl group of $\Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$, and the u_F are rational functions on $\mathfrak{t}$ given by

$$u_F : \mathfrak{t} \ni Y \longmapsto e^{iJ_Y(F)} \int_F \frac{e^{-i\omega}\varrho(Y)}{\chi_{NF}(Y)},$$

[1]By definition, this means that the form $\alpha \in \Lambda^*_c(M)$ is G-invariant, closed, and fulfills $\iota_{\widetilde{X}}\alpha = 0$ for all $X \in \mathfrak{g}$.

[2]Note that, in general, (H^Ω_reg) need not coincide with the principal isotropy type of the G-action on M.

$J_Y(F)$ being the constant value of $J(Y)$ on F and χ_{NF} the equivariant Euler form of the normal bundle NF of F. The definition of the residue operation, given in Definition 2.1, relies on the fact that the Fourier transform of u_F is a piecewise polynomial measure. In case that M_{red} is not connected, each of its components has an open stratum which is connected and dense, and one obtains a residue formula for each component of the reduced space with eventually different H^{Ω}_{reg}. Our approach is in many respects similar to the one of Jeffrey, Kirwan et al., but differs from their's in that it is carried out in the symplectic category and is based on the Marle-Guillemin-Sternberg local normal form and discretized blow-ups. It represents a new proof of their results in the special case of basic differential forms, and extends them to general symplectic manifolds. For a detailed introduction into the problem, historical remarks, and references to the literature, we refer the reader to [21].

We plan to generalize the result presented here from basic differential forms to arbitrary equivariantly closed differential forms in a forthcoming paper. Apart from that, it is likely that our results might be generalized to the Poission category, or at least to tubewise Hamiltonian, or general symplectic actions using the optimal momentum map [20]. Also, instead of considering the action of a compact group, one might be able to extend our results to proper actions.

2. Localization in equivariant cohomology

Let M be a $2n$-dimensional symplectic manifold with symplectic form ω and Riemannian metric g. Then M is orientable, which is equivalent to the fact that $\omega^n/n!$ defines a volume form on M called the *Liouville form*. Let us further define a bundle morphism $\mathcal{I}: TM \to TM$ by setting

$$g_p(\mathcal{I}\mathfrak{X}, \mathfrak{Y}) = \omega_p(\mathfrak{X}, \mathfrak{Y}), \qquad \mathfrak{X}, \mathfrak{Y} \in T_pM,$$

and assume that $\mathcal{I}$ is normed in such a way that $\mathcal{I}^2 = -1$, which defines $\mathcal{I}$ uniquely. Then $(M, \mathcal{I}, g)$ constitutes an almost-Kähler manifold. If $\mathcal{I}$ is integrable, $(M, \mathcal{I}, g)$ becomes a Kähler manifold. Next, assume that M carries a Hamiltonian action of a compact Lie group G of dimension d, and denote the corresponding Kostant-Souriau momentum map by

$$J: M \to \mathfrak{g}^*, \quad J(p)(X) = J_X(p),$$

which is characterized by the property

$$dJ_X + \iota_{\widetilde{X}}\omega = 0 \qquad \forall\, X \in \mathfrak{g}, \tag{2.1}$$

where $\widetilde{X}$ denotes the fundamental vector field on M associated to X.

Remark 2.1. At this point we should mention that there are two sign conventions relevant to this paper that vary in the literature. The first is whether one requires $dJ_X + \iota_{\widetilde{X}}\omega = 0$, as we do, or $dJ_X - \iota_{\widetilde{X}}\omega = 0$, which corresponds to replacing the moment map J by $-J$. The second convention concerns the question whether one defines the differential in the complex of equivariant differential forms by $D(\alpha)(X) = d(\alpha(X)) - \iota_{\widetilde{X}}(\alpha)(X)$, as we do, or by $D(\alpha)(X) = d(\alpha(X)) + \iota_{\widetilde{X}}(\alpha)(X)$, as in [1]. Depending on which sign conventions one uses, the equivariantly closed extension of the symplectic form ω is either $J - \omega$, as in this paper, or $J + \omega$, as in [12]. There are various other conventions leading to different constants in (1.3), compare [13, footnotes on p. 125].

In what follows, we assume that $\mathfrak{g}$ is endowed with an $\mathrm{Ad}\,(G)$-invariant inner product, which allows us to identify $\mathfrak{g}^*$ with $\mathfrak{g}$. Let further dX and $d\xi$ be corresponding measures on $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively, and denote by

$$\mathcal{F}_{\mathfrak{g}}: \mathcal{S}(\mathfrak{g}^*) \to \mathcal{S}(\mathfrak{g}), \qquad \mathcal{F}_{\mathfrak{g}}: \mathcal{S}'(\mathfrak{g}) \to \mathcal{S}'(g^*)$$

the $\mathfrak{g}$-Fourier transform on the Schwartz space and the space of tempered distributions, respectively. To relate the equivariant cohomology $H^*_G(M)$ of M to the cohomology of the symplectic quotient

$$M_{\text{red}} := \Omega_0/G, \qquad \Omega_\eta := J^{-1}(\eta)$$

one considers the map

$$X \quad \longmapsto \quad L_\alpha(X) := \int_M e^{iJ_X}\alpha, \qquad X \in \mathfrak{g}, \qquad \alpha \in \Lambda_c(M),$$

regarded as a tempered distribution in $\mathcal{S}'(\mathfrak{g})$, where $\Lambda_c(M)$ denotes the algebra of differential forms on M of compact support, see [23] and [12]. One is interested in the $\mathfrak{g}$-Fourier transform $\mathcal{F}_\mathfrak{g} L_\alpha$ of L_α, and particularly in its description near $0 \in \mathfrak{g}^*$. For this sake, take an $\mathrm{Ad}^*(G)$-invariant function $\phi \in \mathcal{S}(\mathfrak{g}^*)$ with total integral equal to 1 and the property that its Fourier transform

$$\hat{\phi}(X) = (\mathcal{F}_\mathfrak{g}\phi)(X) = \int_{\mathfrak{g}^*} e^{-i\langle \xi, X\rangle}\phi(\xi)\, d\xi, \qquad \langle \xi, X\rangle = \xi(X), \quad X \in \mathfrak{g},$$

is compactly supported. Then $\phi_\varepsilon(\xi) := \phi(\varepsilon^{-1}\xi)/\varepsilon^d$, $\varepsilon > 0$, constitutes an approximation of the δ-distribution in $\mathfrak{g}^*$ at 0 as $\varepsilon \to 0$, and we consider the limit

$$\lim_{\varepsilon\to 0} \langle \mathcal{F}_\mathfrak{g} L_\alpha, \phi_\varepsilon\rangle = \lim_{\varepsilon\to 0}\int_\mathfrak{g} L_\alpha(X)\hat{\phi}(\varepsilon X)\, dX = \lim_{\varepsilon\to 0}\int_\mathfrak{g}\int_M e^{iJ_X/\varepsilon}\alpha\, \hat{\phi}(X)\frac{dX}{\varepsilon^d}, \tag{2.2}$$

where we took into account that $\hat{\phi}_\varepsilon(X) = \hat{\phi}(\varepsilon X)$. Next, fix a maximal torus $T \subset G$ of dimension d_T with Lie algebra $\mathfrak{t}$, and consider the root space decomposition

$$\mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \bigoplus_{\gamma\in\Delta} \mathfrak{g}_\gamma,$$

where $\Delta = \Delta(\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$ denotes the set of roots of $\mathfrak{g}^\mathbb{C} = \mathfrak{g}\otimes_\mathbb{R}\mathbb{C}$ with respect to $\mathfrak{t}^\mathbb{C} = \mathfrak{t}\otimes_\mathbb{R}\mathbb{C}$, $\mathfrak{g}^\mathbb{C}$ being a reductive Lie algebra over $\mathbb{C}$, and $\mathfrak{g}_\gamma$ are the corresponding root spaces. Since $\dim_\mathbb{C}\mathfrak{g}_\gamma = 1$, the decomposition implies $d - d_T = \dim_\mathbb{R}\mathfrak{g} - \dim_\mathbb{R}\mathfrak{t} = |\Delta|$. Assume that α is such that L_α is $\mathrm{Ad}(G)$-invariant. Using Weyl's integration formula [12, Lemma 3.1], (2.2) can be rewritten as

$$\lim_{\varepsilon\to 0} \langle \mathcal{F}_\mathfrak{g} L_\alpha, \phi_\varepsilon\rangle = \frac{\mathrm{vol}\, G}{|W|\mathrm{vol}\, T}\lim_{\varepsilon\to 0}\int_\mathfrak{t}\left[\int_M e^{iJ_Y}\alpha\right]\hat{\phi}(\varepsilon Y)\Phi^2(Y)dY, \tag{2.3}$$

where $\Phi(Y) = \prod_{\gamma\in\Delta_+}\gamma(Y)$ and Δ_+ is the set of positive roots, while $W = W(\mathfrak{g}^\mathbb{C}, \mathfrak{t}^\mathbb{C})$ denotes the Weyl group. Here $\mathrm{vol}\, G$ and $\mathrm{vol}\, T$ stand for the volumes of G and T with respect to the corresponding volume forms on G and T induced by the invariant inner product on $\mathfrak{g}$ and its restriction to $\mathfrak{t}$, respectively. In what follows, we shall express this limit in terms of the set

$$F^T := \{p \in M : t\cdot p = p \quad \forall\, t \in T\}$$

of fixed points of the underlying T-action. The connected components of F^T are smooth sub-manifolds of possibly different dimensions, and we denote the set of these components by $\mathcal{F}$. Let $F \in \mathcal{F}$ be fixed, and consider the normal bundle NF of F. As can be shown, the real vector bundle NF can be given a complex structure, and splits into a direct sum of two-dimensional real bundles P^F_q, which can be regarded as complex line bundles over F. For each $p \in F$, the fibers $(P^F_q)_p$ are T-invariant, and endowing them with the standard complex structure, the action of $\mathfrak{t}$ can be written as

$$(P^F_q)_p \ni v \mapsto i\lambda^F_q(Y)v \in (P^F_q)_p, \qquad Y \in \mathfrak{t},$$

where the $\lambda^F_q \in \mathfrak{t}^*$ are the weights of the torus action [7]. They do not depend on p.

Now, if ϱ is an equivariantly closed form, $e^{i(J_Y-\omega)}\varrho(Y)$ is equivariantly closed as well, and the integral $L_{e^{-i\omega}\varrho(Y)}(Y)$ can be computed using the localization formula in equivariant cohomology for compactly supported forms, proved independently by Berline and Vergne [3] and Atiyah and Bott [1] at approximately the same time. To apply this formula in our context, recall that an element $Y \in \mathfrak{t}$ is called *regular*, if the set $\{\exp(sY) : s \in \mathbb{R}\}$ is dense in T. The set of regular elements, in the following denoted by $\mathfrak{t}'$, is dense in $\mathfrak{t}$ and

$$\big\{p \in M : \widetilde{Y}_p = 0\big\} = F^T, \qquad Y \in \mathfrak{t}'. \tag{2.4}$$

We then have the following

Proposition 2.2. *Let $\varrho \in \Lambda^*_G(M)_c$ be an equivariantly closed form on M of compact support and $Y \in \mathfrak{t}'$. Then*

$$L_{e^{-i\omega}\varrho(Y)}(Y) = \int_M e^{i(J_Y-\omega)}\varrho(Y) = \sum_{F\in\mathcal{F}} u_F(Y),$$

where the u_F are rational functions on $\mathfrak{t}$ given by

$$u_F : \mathfrak{t} \ni Y \longmapsto e^{iJ_Y(F)} \int_F \frac{e^{-i\omega}\varrho(Y)}{\chi_{NF}(Y)}, \tag{2.5}$$

$J_Y(F)$ being the constant value of J_Y on F, NF denotes the normal bundle of F, which has been endowed with an orientation compatible with the one of F, and χ_{NF} is the equivariant Euler form of the normal bundle.

Proof. This is a direct consequence of the localization formula, see [1, 12], which generalizes directly from equivariant cohomology on compact manifolds to compactly supported equivariant cohomology. The definition of the equivariant Euler form varies in the literature (cf. [2], [21, Corollary 1]). □

In the last proposition, the equivariant Euler form is given by

$$\chi_{NF}(Y) = \prod_q (c_1(P_q^F) + \lambda_q^F(Y)),$$

where $c_1(P_q^F) \in H^2(F)$ denotes the first Chern class of the complex line bundle P_q^F. Thus,

$$\frac{1}{\chi_{NF}(Y)} = \frac{1}{\prod_q \lambda_q^F(Y)} \prod_q \Big(1 + \frac{c_1(P_q^F)}{\lambda_q^F(Y)}\Big)^{-1} = \frac{1}{\prod_q \lambda_q^F(Y)} \prod_q \sum_{0\leq r_q} (-1)^{r_q} \Big(\frac{c_1(P_q^F)}{\lambda_q^F(Y)}\Big)^{r_q}.$$

Note that the sum in the last expression is finite, since $c_1(P_q^F)/\lambda_q^F(Y)$ is nilpotent. Consequently, the inverse makes sense. We would like to compute (2.3) using Proposition 2.2, but since the rational functions (2.5) are not locally integrable on $\mathfrak{t}$, one cannot proceed directly. Instead note that, since Φ^2 and $\hat\phi$ have analytic continuations to $\mathfrak{t}^{\mathbb{C}}$, Cauchy's integral theorem yields for arbitrary $Z \in \mathfrak{t}$

$$\int_{\mathfrak{t}} \left[\int_M e^{i(J_Y - \omega)}\varrho(Y)\right] (\hat\phi_\varepsilon \Phi^2)(Y)dY = \int_{\mathfrak{t}} \left[\int_M e^{i(J_{Y+iZ} - \omega)}\varrho(Y+iZ)\right] (\hat\phi_\varepsilon \Phi^2)(Y+iZ)dY.$$

Here we took into account that by the Theorem of Paley-Wiener-Schwartz [10, Theorem 7.3.1] $\hat\phi_\varepsilon(Y + iZ)$ is rapidly falling in Y. Let now Λ be a proper cone in the complement of all the hyperplanes $\{Y \in \mathfrak{t} : \lambda_q^F(Y) = 0\}$, so that $Y \in \Lambda$ necessarily implies $\lambda_q^F(Y) \neq 0$ for all q and F. By the foregoing considerations, u_F defines a holomorphic function on $\mathfrak{t}+i\Lambda$, and for arbitrary compact sets $M \subset \operatorname{Int}\Lambda$, there is an estimate of the form

$$|u_F(\zeta)| \leq C(1 + |\zeta|)^N, \qquad \zeta = Y + iZ, \quad \operatorname{Im}\zeta \in M,$$

for some $N \in \mathbb{N}$. The functions $u_F\Phi^k$, $k = 0, 1, 2, \dots$, are holomorphic on $\mathfrak{t} + i\Lambda$, too, and satisfy similar bounds. Then, by the generalized Paley-Wiener-Schwartz theorem [10, Theorem 7.4.2], there exists for each k a distribution $U_F^{\Phi^k} \in \mathcal{D}'(\mathfrak{t}^*)$ such that

$$e^{-\langle\cdot,Z\rangle} U_F^{\Phi^k} \in \mathcal{S}'(\mathfrak{t}^*), \qquad \mathcal{F}_{\mathfrak{t}}^{-1}(e^{-\langle\cdot,Z\rangle} U_F^{\Phi^k}) = (u_F\Phi^k)(\cdot + iZ), \qquad Z \in \Lambda.$$

We therefore obtain with Proposition 2.2 for arbitrary $Z \in \Lambda$ and $\eta \in \mathfrak{t}^*$ the equality

$$\begin{aligned}
\int_{\mathfrak{t}} \left[\int_M e^{i(J_Y - \omega)}\varrho(Y)\right] (e^{-i\langle\eta,\cdot\rangle}\hat\phi_\varepsilon \Phi^2)(Y)dY &= \sum_{F\in\mathcal{F}} \Big\langle (u_F\Phi^2)(\cdot + iZ), (e^{-i\langle\eta,\cdot\rangle}\hat\phi_\varepsilon)(\cdot + iZ)\Big\rangle \\
&= \sum_{F\in\mathcal{F}} \Big\langle e^{-\langle\cdot,Z\rangle} U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}\big((e^{-i\langle\eta,\cdot\rangle}\hat\phi_\varepsilon)(\cdot + iZ)\big)\Big\rangle \\
&= \sum_{F\in\mathcal{F}} \Big\langle U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}\big(e^{-i\langle\eta,\cdot\rangle}\hat\phi_\varepsilon\big)\Big\rangle,
\end{aligned}$$

and with (2.2) and (2.3) we arrive at

$$\lim_{\varepsilon\to 0}\lim_{t\to 0} \int_{\mathfrak{g}}\int_M e^{i(J-t\eta)(X)/\varepsilon} e^{-i\omega}\varrho(X/\varepsilon)\hat\phi(X)\frac{dX}{\varepsilon^d} = \frac{\operatorname{vol} G}{|W|\operatorname{vol} T} \lim_{\varepsilon\to 0} \sum_{F\in\mathcal{F}} \Big\langle U_F^{\Phi^2}, \mathcal{F}_{\mathfrak{t}}^{-1}\big(\hat\phi_\varepsilon\big)\Big\rangle. \tag{2.6}$$

In order to further describe the distributions $U_F^{\Phi^k}$, note that the functions $u_F \Phi^k$ are given by a linear combination of terms of the form

$$\frac{e^{iJ_Y(F)}}{\Pi_q \lambda_q^F(Y)^{r_q}} P(Y), \qquad P \in \mathbb{C}[\mathfrak{t}^*].$$

The crucial observation is now that, due to this fact, the $u_F \Phi^k$ are tempered distributions whose $\mathfrak{t}$-Fourier transforms are *piecewise polynomial* measures [12, Proposition 3.6]. By the continuity of the Fourier transform in $\mathcal{S}'$ we therefore have

$$\mathcal{F}_\mathfrak{t}(u_F\Phi^k) = \mathcal{F}_\mathfrak{t}\Big(\lim_{t\to 0} u_F \Phi^k(\cdot + itZ)\Big) = \lim_{t \to 0} \mathcal{F}_\mathfrak{t}(u_F\Phi^k(\cdot + itZ)) = \lim_{t\to 0} e^{-\langle \cdot, tZ\rangle} U_F^{\Phi^k} = U_F^{\Phi^k}.$$

Thus, $U_F^{\Phi^k} \in \mathcal{S}'(\mathfrak{t}^*)$ is the $\mathfrak{t}$-Fourier transform of $u_F\Phi^k$, and, in particular, a piecewise polynomial measure. Since by (2.6) one is interested in the behavior of $U_F^{\Phi^k}$ near the orgin, one is led to the following

Definition 2.1. *Let $\eta \in \mathfrak{t}^*$ be such that for all $F \in \mathcal{F}$ the Fourier transforms $U_F^{\Phi^k}$ are smooth on the segment $t\eta$, $t \in (0, \delta)$. We then define the so-called* residues

$$\operatorname{Res}^{\Lambda,\eta}(u_F \Phi^k) := \lim_{t \to 0} U_F^{\Phi^k}(t\eta).$$

Note that the limit defining $\operatorname{Res}^{\Lambda,\eta}(u_F\Phi^k)$ certainly exists, but in general for $\eta \neq 0$ it does depend on $\eta/|\eta|$ (and Λ) as $U_F^{\Phi^k}$ need not be continuous at the origin. Furthermore, for arbitrary $Z \in \Lambda$,

$$\begin{aligned}
\operatorname{Res}^{\Lambda,\eta}(u_F\Phi^k) &= \lim_{t\to 0}\lim_{\varepsilon \to 0} \int_{\mathfrak{t}^*} U_F^{\Phi^k}(\xi) \mathcal{F}_\mathfrak{t}^{-1}(e^{-i\langle t\eta, \cdot\rangle}\hat{\phi}_\varepsilon)(\xi)\, d\xi \\
&= \lim_{t\to 0}\lim_{\varepsilon \to 0} \Big\langle \mathcal{F}_\mathfrak{t}^{-1}\big(U_F^{\Phi^k} e^{-\langle \cdot, Z\rangle}\big), \big(e^{-i\langle t\eta,\cdot\rangle}\hat{\phi}_\varepsilon\big)(\cdot + iZ)\Big\rangle \\
&= \lim_{t\to 0}\lim_{\varepsilon \to 0} \int_\mathfrak{t} (u_F\Phi^k)(Y + iZ) e^{-i\langle t\eta, Y + iZ\rangle}\hat{\phi}_\varepsilon(Y + iZ) dY,
\end{aligned}$$

$\mathcal{F}_\mathfrak{t}^{-1}(e^{-i\langle t\eta,\cdot\rangle}\hat{\phi}_\varepsilon)$ being an approximation of the δ-distribution at $\eta \in \mathfrak{t}^*$, in concordance with the definition of the residues in [12, Section 8]. Due to Proposition 2.2 this implies

$$\sum_{F\in\mathcal{F}} \operatorname{Res}^{\Lambda,\eta}(u_F\Phi^k) = \lim_{t\to 0}\lim_{\varepsilon\to 0} \int_\mathfrak{t} \Big[\int_M e^{i(J - t\eta)(Y)} e^{-i\omega} \varrho(Y)\Big] \Phi^k(Y)\hat{\phi}(\varepsilon Y)\, dY, \tag{2.7}$$

and applying Weyl's integration formula to (2.7) we obtain

Proposition 2.3. *Let ϱ be an equivariantly closed differential form on M of compact support. Then*

$$\sum_{F\in\mathcal{F}} \operatorname{Res}^{\Lambda,\eta}(u_F\Phi^2) = \lim_{t\to 0}\lim_{\varepsilon\to 0} \frac{|W|\operatorname{vol} T}{\varepsilon^d \operatorname{vol} G}\int_\mathfrak{g}\int_M e^{i(J-t\eta)(X)/\varepsilon} e^{-i\omega}\varrho(X/\varepsilon)\hat{\phi}(X)\, dX. \tag{2.8}$$

□

The central problem consists therefore in the computation of the limits in (2.6) and (2.8), and their exchangeability, leading us to a systematic study of the asymptotic behavior of integrals of the form

$$I_\eta(\mu) = \int_\mathfrak{g}\Big[\int_M e^{i\psi_\eta(p,X)/\mu} a(p,X)\, dp\Big]\, dX, \qquad \mu \to 0^+, \tag{2.9}$$

where $\mathfrak{g}$ is the Lie algebra of an arbitrary compact Lie group G, $a \in \mathrm{C}^\infty_\mathrm{c}(M \times \mathfrak{g})$ is an amplitude, $dp = \omega^n/n!$ the Liouville measure on M, and dX an Euclidean measure on $\mathfrak{g}$ given by an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$, while

$$\psi_\eta(p, X) = J(p)(X) - \eta(X), \qquad \eta \in \mathfrak{g}^*. \tag{2.10}$$

This will occupy us in the next sections.

3. The stationary phase theorem and discrete desingularization

In what follows, we shall describe the asymptotic behavior of the integrals $I_\eta(\mu)$ defined in (2.9) by means of the stationary phase principle. To recall the latter in the context of vector bundles, let M be an n-dimensional, oriented manifold, and $\pi : E \to M$ an oriented vector bundle of rank l. Let further $\alpha \in \Lambda^q_{cv}(E)$ be a differential form on E with compact support along the fibers, $\tau \in \Lambda_c^{n+l-q}(M)$ a differential form on M of compact support, $\psi \in \mathrm{C}^\infty(E)$, and consider the integral

$$I(\mu) = \int_E e^{i\psi/\mu}(\pi^*\tau) \wedge \alpha, \qquad \mu > 0. \tag{3.1}$$

Let $\iota : M \hookrightarrow E$ denote the zero section. Assume that the critical set of ψ coincides with $\iota(M)$ and that the transversal Hessian of ψ is non-degenerate along $\iota(M)$. Then, for each $N \in \mathbb{N}$, $I(\mu)$ possesses an asymptotic expansion of the form

$$I(\mu) = e^{i\psi_0/\mu} e^{i\frac{\pi}{4}\sigma_\psi} (2\pi\mu)^{\frac{l}{2}} \sum_{j=0}^{N-1} \mu^j Q_j(\psi; \alpha, \tau) + R_N(\mu),$$

where ψ_0 and σ_ψ denote the value of ψ and the signature of the transversal Hessian along $\iota(M)$, respectively. The coefficients Q_j are given by measures supported on $\iota(M)$, and can be computed explicitly, as well as the remainder term $R_N(\mu) = \mathrm{O}(\mu^{l/2+N})$. This remainder term depends on ψ only via second order partial derivatives. In particular, the leading coefficient is given as follows. Let $\{U_j\}_{j\in I}$ be an open covering of M and $\{(U_j, \phi_j)\}_{j\in I}$, $\phi_j : \pi^{-1}(U_j) \xrightarrow{\sim} U_j \times \mathbb{R}^l$, an oriented trivialization of $\pi : E \to M$. Write $s_1, \dots, s_l$ for the fiber coordinates on $E_{|U_j}$ given by ϕ_j. Then,

$$Q_0(\psi; \alpha, \tau) = \int_M \frac{\tau \wedge r(\alpha)}{|\iota^*(\det \mathrm{Hess}_{\mathrm{trans}}\, \psi)|^{1/2}}, \tag{3.2}$$

where the *restriction map* $r : \Lambda^q(E) \to \Lambda^{q-l}(M)$ is locally given by

$$(h_j \circ \phi_j)\,(\pi^*\gamma_j) \wedge ds_{\sigma(1)} \wedge \cdots \wedge ds_{\sigma(p)} \quad \longmapsto \quad \begin{cases} (-1)^{\operatorname{sgn}\sigma}\iota^*(h_j \circ \phi_j)\,\gamma_j, & p = l, \\ 0, & p < l, \end{cases} \tag{3.3}$$

$\gamma_j \in \Lambda^{q-p}(U_j)$, $h_j \in \mathrm{C}^\infty(U_j \times \mathbb{R}^l)$, σ being a permutation in p variables, see [21, Theorem A]. From this, one immediately infers the generalized stationary phase theorem [21, Theorem C].

If the critical set of the phase function is not smooth, the stationary phase principle cannot be applied a priori, and one faces serious difficulties in describing the asymptotic behavior of oscillatory integrals. One possible solution is to first partially resolve the singularities of the critical set, and then apply the stationary phase principle in a suitable resolution space, which was the approach of [21]. To explain it in more detail, consider an oscillatory integral of the form (3.1) in case that the critical set $\mathcal{C} := \iota(M) \subset E = \mathcal{M}$ of the phase function ψ is not clean. Let $\mathcal{I}_\mathcal{C}$ be the ideal sheaf of $\mathcal{C}$, and $\mathcal{I}_\psi := (\psi)$ the ideal sheaf generated by the phase function ψ. The essential idea behind the approach in [21] was to construct a partial monomialization

$$\Pi^*(\mathcal{I}_\psi) \cdot \mathcal{O}_{\tilde{x}, \widetilde{\mathcal{M}}} = \sigma_1^{c_1} \cdots \sigma_k^{c_k}\, \Pi_*^{-1}(\mathcal{I}_\psi) \cdot \mathcal{O}_{\tilde{x}, \widetilde{\mathcal{M}}}, \qquad \tilde{x} \in \widetilde{\mathcal{M}},$$

of the ideal sheaf $\mathcal{I}_\psi$ via a suitable resolution $\Pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ in such a way that the derived ideal $D(\Pi_*^{-1}(\mathcal{I}_\psi))$ is a resolved ideal sheaf. As a consequence, the phase function factorizes locally according to $\psi \circ \Pi \equiv \sigma_1^{c_1} \cdots \sigma_k^{c_k} \cdot \tilde{\psi}^{wk}$. Here $\sigma_1, \dots, \sigma_k$ are local continuous variables near each $\tilde{x} \in \widetilde{\mathcal{M}}$ and c_i are natural numbers. If the corresponding weak transforms $\tilde{\psi}^{wk} := \Pi_*^{-1}(\psi)$ have clean critical sets in the sense of Bott [4], we call such a monomialization *clean*. This enables one to apply the stationary phase theorem in the resolution space $\widetilde{\mathcal{M}}$ to the weak transforms $\tilde{\psi}^{wk}$ with the variables $\sigma_1, \dots, \sigma_k$ as continuous parameters.

In this paper, we use a slightly different approach to compute the integrals $I_\eta(\mu)$. In contrast to the previously described approach, where the stationary phase principle is applied only *once* in the final resolution space $\widetilde{\mathcal{M}}$, we shall implement an iterative desingularization algorithm in which the

stationary phase principle is applied *in each iteration step* to very simple oscillatory integrals of the form (1.1). This is achieved by decomposing the phase function ψ into a linear and a quadratic part by means of the local normal form theorem for the momentum map due to Guillemin-Sternberg [9] and Marle [17], and factorizing the quadratic part by means of certain *discretized blow-ups*. Just as a usual blow-up of the origin in $\mathbb{R}^{2n}$ corresponds to introducing polar coordinates away from the origin, yielding the decomposition

$$\mathbb{R}^{2n} - \{0\} \cong \mathbb{R}_{>0} \times S^{2n-1}, \qquad S^{2n-1} := \left\{x \in \mathbb{R}^{2n} : \|x\| = 1\right\},$$

a discretized blow-up of the origin corresponds to a decomposition

$$\mathbb{R}^{2n} - \{0\} \cong \mathbb{N} \times [\mathscr{B}(R_1) - \mathscr{B}(R_2)], \qquad \mathscr{B}(R_i) := \left\{x \in \mathbb{R}^{2n} : \|x\| < R_i\right\}, \quad R_1 > R_2 > 0,$$

$\mathscr{B}(R_1) - \mathscr{B}(R_2)$ being a spherical shell in $\mathbb{R}^{2n}$. The point is that the interiors of the sherical shells are open subsets of $\mathbb{R}^{2n}$, and as such inherit a symplectic structure from the standard symplectic structure on $\mathbb{R}^{2n}$, whereas the spheres S^{2n-1} cannot carry symplectic structures since they are odd dimensional. Consequently, discretized blow-ups allow us to stay within the symplectic category, in order to be able to apply the local normal form theorem iteratively, and are therefore more suitable for our purposes than usual blow-ups.

4. Equivariant asymptotics and the local normal form of the momentum map

4.1. **Equivariant asymptotics.** We commence now with our study of the asymptotic behavior of the integrals (2.9) by means of the stationary phase principle. To determine the critical set of the phase function $\psi_\eta(p, X)$, let $\{X_1, \dots, X_d\}$ be a basis of $\mathfrak{g}$, and write $X = \sum_{i=1}^d s_i X_i$. Due to the linear dependence of J_X in X,

$$\partial_{s_i} \psi_\eta(p, X) = J_{X_i}(p) - \eta(X_i), \tag{4.1}$$

and because of the non-degeneracy of ω,

$$J_{X,*} = 0 \quad \iff \quad dJ_X = -\iota_{\widetilde{X}}\omega = 0 \quad \iff \quad \widetilde{X} = 0.$$

Hence, the critical set reads

$$\mathrm{Crit}(\psi_\eta) := \{(p, X) \in M \times \mathfrak{g} : \psi_{\eta,*}(p, X) = 0\} = \left\{(p, X) \in \Omega_\eta \times \mathfrak{g} : \widetilde{X}_p = 0\right\}, \tag{4.2}$$

where $\Omega_\eta := J^{-1}(\eta)$ is the η-level of the momentum map. Now, the major difficulty in applying the stationary phase principle in our setting stems from the fact that, due to the singular orbit structure of the underlying group action, Ω_η and, consequently, the considered critical set $\mathrm{Crit}(\psi_\eta)$, are in general singular. In fact, if the G-action on M is not free, Ω_η and the symplectic quotients Ω_η/G_η are no longer smooth for general $\eta \in \mathfrak{g}^*$, where G_η denotes the stabilizer of η under the co-adjoint action. Nevertheless, both Ω_η and Ω_η/G_η have Whitney stratifications into smooth sub-manifolds, see Lerman-Sjamaar [22], and Ortega-Ratiu [20, Theorems 8.3.1 and 8.3.2], which correspond to the stratification of M into orbit types, see Duistermaat-Kolk [6].

Note that from the definition of the momentum map it is clear that the kernel of its derivative is given by

$$\ker J_{*,p} = (\mathfrak{g} \cdot p)^\omega, \qquad p \in M, \tag{4.3}$$

where we denoted the symplectic complement of a subspace $V \subset T_pM$ by V^ω, and wrote $\mathfrak{g} \cdot p = \{\widetilde{X}_p : X \in \mathfrak{g}\}$. Consequently, if $\eta \in J(M)$ is a regular value of the momentum map, Ω_η is a [3] manifold of co-dimension d, and $T_p\Omega_\eta = \ker J_{*,p} = (\mathfrak{g} \cdot p)^\omega$, which is equivalent to the fact that

$$\widetilde{X}_p \neq 0 \qquad \text{for all } p \in \Omega_\eta,\ 0 \neq X \in \mathfrak{g},$$

compare [19, Chapter 8]. The latter condition means that all stabilizers G_p of points $p \in \Omega_\eta$ are finite, and therefore either of regular or exceptional type. In particular, one has $\dim \mathfrak{g} \cdot p = d$ for all $p \in \Omega_\eta$.

[3] not necessarily connected

Thus, if η is a regular value, both Ω_η and $\mathrm{Crit}(\psi_\eta) = \Omega_\eta \times \{0\}$ are differentiable manifolds. In addition, in view of the exact sequence

$$0 \longrightarrow T_p\Omega_\eta \xrightarrow{\iota_{\eta,*}} T_pM \xrightarrow{J_*} T_\eta\mathfrak{g}^* \longrightarrow 0, \qquad p \in \Omega_\eta,$$

where $\iota_\eta : \Omega_\eta \hookrightarrow M$ denotes the inclusion, and the corresponding dual sequence, Ω_η is orientable, M being orientable, compare [16, Chapter XV.6]. One now deduces

Proposition 4.1. *Assume that $\eta \in \mathfrak{g}^*$ is a regular value of the momentum map $J : M \to \mathfrak{g}^*$, and let $I_\eta(\mu)$ be defined as in* (2.9)*. Then, for each $N \in \mathbb{N}$, there exists a constant $C_{N,J,a}$ which is independent of η such that*

$$\Big| I_\eta(\mu) - (2\pi\mu)^d \sum_{j=0}^{N-1} \mu^j Q_j(\psi_\eta, a) \Big| \leq C_{N,J,a}\, \mu^N,$$

where the coefficients Q_j can be expressed explicitly in terms of measures on Ω_η/G.

Proof. As it is shown in the proof of [21, Proposition 2], the critical set of the phase function ψ_η is clean, and the assertion follows directly from the generalized stationary phase theorem [21, Theorem C]. Note that in [21, Proposition 2] the statement is proved with constants $C_{N,\psi_\eta,a}$ that depend a priori on ψ_η. However, looking closely at the proof, one sees that the constants $C_{N,\psi_\eta,a}$ depend in fact only on second order partial derivatives of ψ_η, and since $\psi_\eta(p, X) = J(p)(X) - \eta(X)$ depends on η only via a linear term, no second order derivative of ψ_η depends on η. Thus, the constants $C_{N,\psi_\eta,a}$ can be chosen to be independent of η. $\square$

Now, suppose we *knew* that the singular oscillatory integral $I_0(\mu)$ had an asymptotic expansion of the form

$$I_0(\mu) = (2\pi\mu)^d L_0(\psi_0, a) + \mathrm{o}(\mu^d), \qquad \mu \to 0^+, \tag{4.4}$$

with some explicitly known leading term $L_0(\psi_0, a)$ depending on the chosen amplitude a and the phase ψ_0. In case that $\eta \in \mathfrak{g}^*_{\mathrm{reg}}$, Proposition 4.1 implies the expansion

$$I_\eta(\mu) = (2\pi\mu)^d Q_0(\psi_\eta, a) + \mathrm{O}(\mu^{d+1}), \qquad \mu \to 0^+, \tag{4.5}$$

with explicitly computable leading terms $Q_0(\psi_\eta, a)$, the convergence being uniform in η. By comparing the expansions (4.4) and(4.5), we then would have

$$\lim_{\eta \in \mathfrak{g}^*_{\mathrm{reg}}, \eta \to 0} Q_0(\psi_\eta, a) = L_0(\psi_0, a), \tag{4.6}$$

since $\mathfrak{g}^*_{\mathrm{reg}}$ is dense by Sard's theorem, and $I_\eta(\mu)$ is manifestly continuous in η for arbitrary $\mu > 0$. If therefore $a \in \mathrm{C}^\infty_{\mathrm{c}}(M \times \mathfrak{g})$ is such that, with the notation as in (2.8),

$$I_\eta(\mu) = \frac{|W| \operatorname{vol} T}{\varepsilon^d \operatorname{vol} G} \int_{\mathfrak{g}} \int_M e^{i(J - t\eta)(X)/\varepsilon} e^{-i\omega} \varrho(X/\varepsilon) \hat{\phi}(X)\, dX$$

Proposition 2.3 and (4.6) would allow us to conclude that - *assuming* (4.4) - one would have

$$\sum_{F \in \mathcal{F}} \mathrm{Res}^{\Lambda,\eta}(u_F \Phi^2) = L_0(\psi_0, a), \tag{4.7}$$

yielding the desired residue formula. Thus, we are left with the task to prove an expansion of the form (4.4), with an explicit expression for $L_0(\psi_0, a)$ in terms of the reduced space $M_{\mathrm{red}} = \Omega_0/G$. This amounts to an examination of the asymptotic behavior of the integrals (2.9) in case that $\eta = 0$ is a singular value of the momentum map, in which case $\mathrm{Crit}(\psi_0)$ is singular. From now on, we will simply write

$$\psi \text{ for } \psi_0, \qquad I(\mu) \text{ for } I_0(\mu), \qquad \Omega \text{ for } \Omega_0, \qquad \mathcal{C} \text{ for } \mathcal{C}_0.$$

To begin, recall that Ω has a decomposition into smooth manifolds given by

$$\Omega = \bigcup_{H < G} \Omega_{(H)},$$

where $\Omega_{(H)} := \Omega \cap M_{(H)}$ denotes the union of orbits in Ω of type (H). To this decomposition corresponds a stratification of M_{red} into a union of disjoint symplectic manifolds

$$M_{\mathrm{red}} = \Omega/G = \bigcup_{H<G} \Omega_{(H)}/G,$$

see [22, Theorem 2.1]. Ω and the strata $\Omega_{(H)}$ might not be connected. But as the following lemma shows, the components of $\Omega_{(H)}$ have the same dimension.

Lemma 4.2. *Let $H \subset G$ be a closed subgroup. The co-dimension of the stratum $\Omega_{(H)}$ of Ω is $d - \dim H$, and*

$$T_p\Omega_{(H)} = [T_p(G\cdot p)]^\omega = (\mathfrak{g}\cdot p)^\omega, \qquad p \in \Omega_{(H)}. \tag{4.8}$$

Furthermore, the critical set (4.2) of the phase function $\psi(p,X) = J(p)(X)$ decomposes into smooth manifolds

$$\mathrm{Crit}(\psi)_{(H)} := \big\{(p,X) \in \Omega_{(H)} \times \mathfrak{g} : X \in \mathfrak{g}_p\big\} \tag{4.9}$$

of co-dimension $2(d - \dim H)$, and

$$T_{(p,X)}\mathrm{Crit}(\psi)_{(H)} = \left\{(\mathfrak{X}, w) \in (\mathfrak{g}\cdot p)^\omega \times \mathbb{R}^d : \sum_{i=1}^d w_i(\widetilde{X}_i)_p = [\widetilde{X}, \widetilde{\mathfrak{X}}]_p\right\}, \tag{4.10}$$

where $\widetilde{\mathfrak{X}}$ denotes an extension of $\mathfrak{X}$ to a vector field [4]*. Finally, if $\dim H = 0$, each $p \in \Omega_{(H)}$ is a regular point of J, and both $\Omega_{(H)}$ and $\mathrm{Crit}(\psi)_{(H)} \simeq \Omega_{(H)} \times \{0\}$ are orientable.*

Proof. The proof is a verbatim repetition of the arguments given in [21, Lemma 2], applied to each stratum $\Omega_{(H)}$. Let $p(t)$ be a smooth curve in $\Omega_{(H)}$ and write $\mathfrak{X} = \dot p(t_0) \in T_{p(t_0)}\Omega_{(H)}$. Differentiating the equality $J(p(t))(X) = J_X(p(t)) = 0$ for arbitrary $X \in \mathfrak{g}$ yields

$$\frac{d}{dt} J_X(p(t))_{|t=t_0} = dJ_X(p(t_0)) \circ \dot p(t_0) = -\omega(\widetilde{X}, \mathfrak{X})_{|p(t_0)} = 0,$$

and we obtain (4.8). But then $\dim \mathfrak{g}\cdot p + \dim(\mathfrak{g}\cdot p)^\omega = 2n$ implies that $\mathrm{codim}\Omega_{(H)} = d - \dim H$. Further, since the Lie algebra of G_p is given by $\mathfrak{g}_p = \{X \in \mathfrak{g} : \widetilde{X}_p = 0\}$, (4.9) follows from (4.2). To see (4.10), let $(p(t), X(t))$ be a smooth curve in $\Omega_{(H)} \times \mathfrak{g}$. Writing $X(t) = \sum s_j(t) X_j$ with respect to a basis $\{X_1, \dots, X_d\}$ of $\mathfrak{g}$, one computes for any $f \in \mathrm{C}^\infty(\Omega_{(H)})$

$$\begin{aligned}
\frac{d}{dt}\widetilde{X(t)}_{p(t)} f_{|t=t_0} &= \sum_{j=1}^d \frac{d}{dt}\Big(s_j(t)(\widetilde{X_j})_{p(t)} f\Big)_{|t=t_0} \\
&= \sum_{j=1}^d \dot s_j(t_0)(\widetilde{X_j} f)(p(t_0)) + \sum_{j=1}^d s_j(t_0)\frac{d}{dt}(\widetilde{X_j} f)(p(t))_{|t=t_0}.
\end{aligned}$$

Writing $\mathfrak{X} = \dot p(t_0) \in T_{p(t_0)}\Omega_{(H)}$, one has $\frac{d}{dt}(\widetilde{X_j} f)(p(t))_{|t=t_0} = \widetilde{\mathfrak{X}}_{p(t_0)}(\widetilde{X_j} f)$, so that if $(p(t), X(t))$ is a curve in $\mathrm{Crit}(\psi)_{(H)}$ one obtains

$$\sum_{j=1}^d \dot s_j(t_0)(\widetilde{X_j})_{p(t_0)} f + \sum_{j=1}^d s_j(t_0)[\widetilde{\mathfrak{X}}, \widetilde{X_j}]_{p(t_0)} f = 0,$$

since $\widetilde{X(t)}_{p(t)} = 0$ for all t, and the assertion follows from (4.8). Finally, if $\dim H = 0$, (4.3) implies that each element $p \in \Omega_{(H)}$ is a regular point of J, and because of the exact sequence

$$0 \longrightarrow T_p\Omega_{(H)} \xrightarrow{\iota_*} T_pM \xrightarrow{J_*} T_0\mathfrak{g}^* \longrightarrow 0, \qquad p \in \Omega_{(H)},$$

where $\iota : \Omega_{(H)} \hookrightarrow M$ denotes the inclusion, and the corresponding dual sequence, $\Omega_{(H)}$ is orientable by [16, Chapter XV.6]. $\square$

[4] In the proposition below, we shall actually see that $[\widetilde{X}, \widetilde{\mathfrak{X}}]_p \in \mathfrak{g}\cdot p$ for $X \in \mathfrak{g}_p$ and $\mathfrak{X} \in (\mathfrak{g}\cdot p)^\omega$.

For later use, let us mention the following

Proposition 4.3. *For each subgroup $H \subset G$, the mapping $P : \mathrm{Crit}(\psi)_{(H)} \to \Omega_{(H)}$, $(p, X) \mapsto p$ is a submersion.*

Proof. Again, the proof is a repetition of the arguments given in [21, Proposition 4], applied to each stratum $\Omega_{(H)}$. Thus, let $p \in \Omega_{(H)}$ and $X \in \mathfrak{g}_p$. We show that $[\widetilde{\mathfrak{X}}, \widetilde{X}]_p \in \mathfrak{g} \cdot p$ for all $\mathfrak{X} \in T_p\Omega_{(H)}$. To begin, note that $\pi_G : \Omega_{(H)} \to \Omega_{(H)}/G$ is a submersion and a principal fiber bundle with $\ker(\pi_G)_{*,p} = \mathfrak{g} \cdot p$ [20, Theorem 8.1.1]. If therefore $p(s) \in \Omega_{(H)}$ denotes a curve with $p(0) = p$, $\dot p(0) = \mathfrak{X}$, and $g \in G_p$, differentiation of $\pi_G(g \cdot p(s)) = \pi_G(p(s))$ yields $\mathfrak{X} - g_{*,p}(\mathfrak{X}) \in \ker(\pi_G)_{*,p} = \mathfrak{g} \cdot p$. Consequently,

$$\frac{d}{dt}(e^{-tX})_{*,p}\mathfrak{X}_{|t=0} = \lim_{t \to 0} t^{-1}\big[(e^{-tX})_{*,p}\mathfrak{X} - \mathfrak{X}\big] \in \mathfrak{g} \cdot p, \tag{4.11}$$

where we made the identification $T_{\mathfrak{X}}(T_p\Omega_{(H)}) \simeq T_p\Omega_{(H)}$. Now, for arbitrary $Y \in \mathfrak{g}$ [20, Proposition 4.2.2],

$$\omega_p([\widetilde{\mathfrak{X}}, \widetilde{X}], \widetilde{Y}) = -\omega_p([\widetilde{X}, \widetilde{Y}], \widetilde{\mathfrak{X}}) - \omega_p([\widetilde{Y}, \widetilde{\mathfrak{X}}], \widetilde{X}) = 0,$$

since $\widetilde{X}_p = 0$, and $\widetilde{\mathfrak{X}}_p = \mathfrak{X} \in (\mathfrak{g} \cdot p)^\omega$. Hence, $[\widetilde{\mathfrak{X}}, \widetilde{X}]_p \in (\mathfrak{g} \cdot p)^\omega$. Furthermore, for arbitrary $f \in \mathrm{C}^\infty(M)$,

$$[\widetilde{\mathfrak{X}}, \widetilde{X}]_p f = \widetilde{\mathfrak{X}}_p(\widetilde{X} f) = \frac{d}{ds}(\widetilde{X} f)(p(s))_{|s=0} = \frac{d}{dt}\left(\frac{d}{ds} f(e^{-tX} \cdot p(s))_{|s=0}\right)_{|t=0} = \frac{d}{dt}\big(\widetilde{(e^{-tX})_{*,p}\mathfrak{X}}_{|t=0}\big)_p f,$$

so that with (4.11)

$$[\widetilde{\mathfrak{X}}, \widetilde{X}]_p = \frac{d}{dt}(e^{-tX})_{*,p}\mathfrak{X}_{|t=0} \in \mathfrak{g} \cdot p. \tag{4.12}$$

The previous lemma then implies that $P_{*,(p,X)} : T_{(p,X)}\mathrm{Crit}(\psi)_{(H)} \to T_p\Omega_{(H)}$, $(\mathfrak{X}, w) \mapsto \mathfrak{X}$ is a surjection, and the assertion follows. □

Remark 4.4. Note that for $p \in \Omega_{(H)}$, and $X \in \mathfrak{g}_p$, the previous proposition implies that the Lie derivative defines a homomorphism

$$L_X : \mathfrak{g} \cdot p \ni \mathfrak{X} \longmapsto \mathcal{L}_{\widetilde{X}}(\widetilde{\mathfrak{X}})_p = [\widetilde{X}, \widetilde{\mathfrak{X}}]_p \in \mathfrak{g} \cdot p. \tag{4.13}$$

Next, we observe that by [22, Theorem 5.9 and Remark 5.10], each connected component of the symplectic quotient Ω/G has a unique open stratum that is connected and dense. For simplicity, let us assume that $M_{\mathrm{red}} = \Omega/G$ is connected, and denote its open, connected and dense stratum by

$$\mathrm{Reg}\, M_{\mathrm{red}} := \Omega_{(H^\Omega_{\mathrm{reg}})}/G.$$

Since the projection $\pi : \Omega \to \Omega/G$ is a continuous and open map, $\mathrm{Reg}\,\Omega := \Omega_{(H^\Omega_{\mathrm{reg}})}$ is open, connected, and dense in Ω. In what follows we shall write

$$\mathcal{C} := \mathrm{Crit}(\psi), \qquad \mathrm{Reg}\,\mathcal{C} := \mathrm{Crit}(\psi)_{(H^\Omega_{\mathrm{reg}})}, \tag{4.14}$$

and call $\mathrm{Reg}\, M_{\mathrm{red}}$, $\mathrm{Reg}\,\Omega$, and $\mathrm{Reg}\,\mathcal{C}$ the *regular stratum* of M_{red}, Ω, and $\mathcal{C}$, respectively, and $(H^\Omega_{\mathrm{reg}})$ the *regular isotropy type.* We now have the following

Lemma 4.5. *The stratum $\mathrm{Reg}\,\mathcal{C}$ of the critical set $\mathcal{C}$ is clean. Furthermore, for $(p, X) \in \mathrm{Reg}\,\mathcal{C}$ the transversal Hessian is given by*

$$\det \mathrm{Hess}\, \psi(p, X)_{|N_{(p,X)}\mathrm{Reg}\,\mathcal{C}} = \det\,(\Xi - L_X \circ L_X)_{|\mathfrak{g}\cdot p},$$

where $L_X : \mathfrak{g} \cdot p \to \mathfrak{g} \cdot p$ denotes the linear mapping (4.13) *given by the Lie derivative, and Ξ the linear transformation on $\mathfrak{g} \cdot p$ defined in* (4.15).

Proof. The proof is a direct generalization of the proof of [21, Lemma 7] to our context. By Lemma 4.2, $\mathrm{Reg}\,\mathcal{C}$ is a differentiable manifold of co-dimension $2(d - \dim H^\Omega_{\mathrm{reg}})$. Its tangent bundle is given by (4.10). By definition, the Hessian of ψ at $(p, X) \in \mathrm{Reg}\,\mathcal{C}$ is given by the symmetric bilinear form

$$\mathrm{Hess}\, \psi : T_{(p,X)}(M \times \mathfrak{g}) \times T_{(p,X)}(M \times \mathfrak{g}) \to \mathbb{C}, \qquad (v_1, v_2) \mapsto \tilde v_1(\tilde v_2(\psi))(p, X).$$

Let $\{\widetilde{\mathfrak{X}}_1,\dots,\widetilde{\mathfrak{X}}_{2n}\}$ be a local orthonormal frame in TM and $\{e_1,\dots,e_d\}$ the standard basis in $\mathbb{R}^d$ corresponding to an orthonormal basis $\{A_1,\dots,A_d\}$ of $\mathfrak{g}$ such that $\left\{A_1,\dots,A_{d-\dim H^\Omega_{\mathrm{reg}}}\right\}$ is a basis of $\mathfrak{g}_p^\perp$. In the basis

$$((\widetilde{\mathfrak{X}}_i)_p;0),\qquad (0;e_j),\qquad i=1,\dots,2n,\quad j=1,\dots,d,$$

of $T_{(p,X)}(M\times\mathfrak{g})=T_pM\times\mathbb{R}^d$, $\operatorname{Hess}\psi$ is given by the matrix

$$\mathcal{A}:=\begin{pmatrix}\omega_p([\widetilde{X},\widetilde{\mathfrak{X}}_i],\widetilde{\mathfrak{X}}_j) & -\omega_p(\widetilde{A}_j,\widetilde{\mathfrak{X}}_i)\\ -\omega_p(\widetilde{A}_i,\widetilde{\mathfrak{X}}_j) & 0\end{pmatrix}=\begin{pmatrix}\mathcal{I}L_X & -g_p(\mathcal{I}\widetilde{A}_j,\widetilde{\mathfrak{X}}_i)\\ -g_p(\mathcal{I}\widetilde{A}_i,\widetilde{\mathfrak{X}}_j) & 0\end{pmatrix},$$

where $\mathcal{I}:TM\to TM$ denotes the bundle homomorphism introduced in Section 2. Indeed, $\widetilde{\mathfrak{X}}_i(J_X)=dJ_X(\widetilde{\mathfrak{X}}_i)=-\iota_{\widetilde{X}}\omega(\widetilde{\mathfrak{X}}_i)$, and $(\widetilde{\mathfrak{X}}_i)_p(\omega(\widetilde{X},\widetilde{\mathfrak{X}}_j))=-\omega_p([\widetilde{X},\widetilde{\mathfrak{X}}_i],\widetilde{\mathfrak{X}}_j)$, since $\widetilde{X}_p=0$. where $L_X:T_pM\to T_pM,\mathfrak{X}\mapsto[\widetilde{X},\widetilde{\mathfrak{X}}]_p$ denotes the linear transformation induced by the Lie derivative, and restricts to a map on $\mathfrak{g}\cdot p$ by Remark 4.4. In order to compute the transversal Hessian, let $\{B_1,\dots,B_{d-\dim H^\Omega_{\mathrm{reg}}}\}$ be another basis of $\mathfrak{g}_p^\perp$ such that $\{(\widetilde{B}_1)_p,\dots,(\widetilde{B}_{d-\dim H^\Omega_{\mathrm{reg}}})_p\}$ is an orthonormal basis of $\mathfrak{g}\cdot p$, and recall that by (4.8) we have $T_p\Omega_{(H^\Omega_{\mathrm{reg}})}=(\mathfrak{g}\cdot p)^\omega$. Taking into account (4.10) and $\mathfrak{g}\cdot p\subset(\mathfrak{g}\cdot p)^\omega$ one sees that

$$\mathcal{B}_k=(\mathcal{J}(\widetilde{B}_k)_p;0),\qquad \mathcal{B}'_k=(L_X(\widetilde{B}_k)_p;g_p(\widetilde{A}_1,\widetilde{B}_k),\dots,g_p(\widetilde{A}_{d-H^\Omega_{\mathrm{reg}}},\widetilde{B}_k),0,\dots,0),$$

where $k=1,\dots,d-\dim H^\Omega_{\mathrm{reg}}$, constitutes a basis of $N_{(p,X)}\mathrm{Reg}\,\mathcal{C}$ with $\langle\mathcal{B}_k,\mathcal{B}_l\rangle=\delta_{kl}$, $\mathcal{B}_k\perp\mathcal{B}'_l$, and $\langle\mathcal{B}'_k,\mathcal{B}'_l\rangle=(\Xi+L_XL_X)_{kl}$, where Ξ is defined

$$\Xi:\mathfrak{g}\cdot p\longrightarrow\mathfrak{g}\cdot p:\mathfrak{X}\mapsto\sum_{j=1}^d g_p(\mathfrak{X},\widetilde{A}_j)(\widetilde{A}_j)_p. \tag{4.15}$$

One now computes

$$\begin{aligned}\mathcal{A}(\mathcal{B}_k)=&\Big(\mathcal{J}L_X\mathcal{J}(\widetilde{B}_k)_p;-\sum_{j=1}^{2n}g_p(\mathcal{J}\widetilde{A}_1,\widetilde{\mathfrak{X}}_j)g_p(\mathcal{J}\widetilde{B}_k,\widetilde{\mathfrak{X}}_j),\dots\Big)\\ =&(-L_X(\widetilde{B}_k)_p;-g_p(\mathcal{J}\widetilde{A}_1,\mathcal{J}\widetilde{B}_k),\dots,-g_p(\mathcal{J}\widetilde{A}_{d-\dim H^\Omega_{\mathrm{reg}}},\mathcal{J}\widetilde{B}_k),0,\dots,0)=-\mathcal{B}'_k,\\ \mathcal{A}(\mathcal{B}'_k)=&\Big(\mathcal{J}L_XL_X(\widetilde{B}_k)_p-\Big(\sum_{j=1}^{d-\dim H^\Omega_{\mathrm{reg}}}g_p(\mathcal{J}\widetilde{A}_j,\widetilde{\mathfrak{X}}_1)g_p(\widetilde{A}_j,\widetilde{B}_k),\dots\Big);\\ &-\sum_{j=1}^{2n}g_p(\mathcal{J}\widetilde{A}_1,\widetilde{\mathfrak{X}}_j)g_p(L_X(\widetilde{B}_k)_p,\widetilde{\mathfrak{X}}_j),\dots\Big)=(\mathcal{J}L_XL_X(\widetilde{B}_k)_p+(g_p(\Xi(\widetilde{B}_k)_p,\mathcal{J}\widetilde{\mathfrak{X}}_1),\dots);\\ &-g_p(\mathcal{J}\widetilde{A}_1,L_X(\widetilde{B}_k)_p),\dots).\end{aligned}$$

Since L_X defines an endomorphism of $\mathfrak{g}\cdot p$ and $\mathfrak{g}\cdot p\subset(\mathfrak{g}\cdot p)^\omega$ we have $g_p(\mathcal{J}\widetilde{A}_1,L_X(\widetilde{B}_k)_p)=\omega_p(\widetilde{A}_1,L_X(\widetilde{B}_k)_p)=0$. Furthermore, the $\{\mathcal{J}(\widetilde{B}_1)_p,\dots,\mathcal{J}(\widetilde{B}_{d-\dim H^\Omega_{\mathrm{reg}}})_p\}$ form an orthonormal basis of $\mathcal{J}(\mathfrak{g}\cdot p)$, and we obtain

$$\mathcal{A}(\mathcal{B}'_k)=(\mathcal{J}(L_XL_X-\Xi)(\widetilde{B}_k)_p;0)=\sum_{j=1}^{d-\dim H^\Omega_{\mathrm{reg}}}g_p(\mathcal{J}(L_XL_X-\Xi)(\widetilde{B}_k)_p,\mathcal{J}(\widetilde{B}_j)_p)\,\mathcal{B}_j.$$

Taking all together, one sees that the transversal Hessian $\operatorname{Hess}\psi(p,X)_{|N_{(p,X)}\mathrm{Reg}\,\mathcal{C}}$ is given by the matrix

$$\begin{pmatrix}0 & -\mathbf{1}_{d-\dim H^\Omega_{\mathrm{reg}}}\\ (L_XL_X-\Xi)_{|\mathfrak{g}\cdot p} & 0\end{pmatrix},$$

and the assertion follows. □

Remark 4.6. The arguments given in the previous lemma apply to any stratum of $\mathcal{C}$. Thus, each of them taken by itself is clean.

To formulate the main result of this sub-section, for any G-orbit $\mathcal{O} \subset M$ and any continuous function f defined on $\mathcal{O}$ write

$$\widetilde{f}(\mathcal{O}) := \fint_{\mathcal{O}} f(p)\, d\mathcal{O}(p) := \frac{1}{\operatorname{vol} G} \int_G f(g \cdot p_{\mathcal{O}})\, dg = \frac{1}{\operatorname{vol} \mathcal{O}} \int_{\mathcal{O}} f(p)\, d\mathcal{O}(p), \tag{4.16}$$

where dg is any Haar measure on G, $p_{\mathcal{O}} \in \mathcal{O}$ an arbitrary point, and $d\mathcal{O}$ is an arbitrary G-invariant Riemannian measure on the orbit $\mathcal{O}$ if $\dim \mathcal{O} > 0$, and the discrete counting measure otherwise. We then have

Proposition 4.7. *Let $I(\mu)$ be the oscillatory integral defined in* (2.9) *with $\eta = 0$, and assume that the support of $a(p, X)$ does only intersect the stratum $\operatorname{Reg} \mathcal{C}$ of $\mathcal{C}$. Then, for each $N \in \mathbb{N}$, there exists a constant $C_{N,\psi,a}$ such that*

$$\Big| I(\mu) - (2\pi\mu)^{d - \dim H^{\Omega}_{\mathrm{reg}}} \sum_{j=0}^{N-1} \mu^j Q_j(\psi, a) \Big| \leq C_{N,\psi,a}\, \mu^N \qquad \forall\, \mu > 0,$$

where the coefficients Q_j are given explicitly. In particular, the leading coefficient is given by

$$\begin{aligned} Q_0(\psi, a) &= \int_{\operatorname{Reg} \mathcal{C}} \frac{a(p, X)}{|\det \operatorname{Hess} \psi(p, X)_{|N_{(p,X)} \operatorname{Reg} \mathcal{C}}|^{1/2}}\, d(\operatorname{Reg} \mathcal{C})(p, X) \\ &= \frac{\operatorname{vol} G}{\operatorname{vol} H^{\Omega}_{\mathrm{reg}}} \int_{\operatorname{Reg} \Omega} \left[\int_{\mathfrak{g}_p} a(p, X)\, dX \right] \frac{d(\operatorname{Reg} \Omega)(p)}{\operatorname{vol} \mathcal{O}_p} \\ &= \frac{\operatorname{vol} G}{\operatorname{vol} H^{\Omega}_{\mathrm{reg}}} \int_{\operatorname{Reg} M_{\mathrm{red}}} \fint_{\mathcal{O}} \int_{\mathfrak{g}_p} a(p, X)\, dX\, d\mathcal{O}(p)\, d(\operatorname{Reg} M_{\mathrm{red}})(\mathcal{O}), \end{aligned}$$

where $d(\operatorname{Reg} \Omega)$ denotes the Riemannian volume measure induced on $\operatorname{Reg} \Omega$ given by some G-invariant Riemannian metric on M, and $\operatorname{vol} \mathcal{O}_p$ the corresponding Riemannian volume of the orbit through p, while $d(\operatorname{Reg} M_{\mathrm{red}})$ is the symplectic volume form on $\operatorname{Reg} M_{\mathrm{red}}$.

Proof. This is an immediate consequence of the generalized stationary phase theorem and Lemma 4.5, together with the arguments given in the proof of [21, Proposition 6], adapted to our context. Another computation of the leading term will be given in Section 7.1. □

Thus, we are left with the task of examining the integrals $I(\mu)$ when the support of the amplitude $a(p, X)$ does not only intersect the regular stratum $\operatorname{Reg} \mathcal{C}$, but also other strata. As already explained in Section 3, our strategy will be to develop an iterative desingularization process consisting in a series of discretized blow-ups. If M is the co-tangent bundle of a G-manifold, a partial desingularization was carried out in [21]. Partial desingularizations of the zero level set $\Omega = J^{-1}(0)$ of the momentum map and the symplectic quotient Ω/G have been obtained by Meinrenken-Sjamaar [18] for compact symplectic manifolds with a Hamiltonian compact Lie group action by performing blow-ups along minimal symplectic sub-orbifolds containing the strata of maximal depth in Ω. In the context of geometric invariant-theoretic quotients, partial desingularizations were studied in [15] and [11].

To close this section, let us mention that the residues in (2.7) can also be expressed via symplectic reduction with respect to the action of a maximal torus $T \subset G$. Indeed, let $\eta \in \mathfrak{g}^*$ be a regular value of $J : M \to \mathfrak{g}^*$, $\alpha \in \Lambda_c(M)$, $\theta \in S^r(\mathfrak{g}^*)$, and $\phi \in \mathcal{S}(\mathfrak{g}^*)$ an $\operatorname{Ad}^*(G)$-invariant function with total integral equal to 1 and compactly supported Fourier transform. Then

$$\lim_{\varepsilon \to 0} \int_{\mathfrak{g}} \left[\int_M e^{i(J-\eta)(X)} \alpha \right] \theta(X) \hat{\phi}(\varepsilon X)\, dX = \frac{(2\pi)^d \operatorname{vol} G}{|H^{\Omega_\eta}_{\mathrm{reg}}|} \int_{J^{-1}(\eta)/G} (\pi_\eta^*)^{-1} \circ \iota_\eta^*(\mathfrak{L}) \tag{4.17}$$

for some form $\mathfrak{L} \in \Lambda_c(M)$ explicitly given in terms of J, α and θ, where $\iota_\eta : J^{-1}(\eta) \hookrightarrow M$ is the inclusion and $\pi_\eta : J^{-1}(\eta) \to J^{-1}(\eta)/G$ the canonical projection, while $H^{\Omega_\eta}_{\mathrm{reg}}$ denotes the isotropy group of a generic point in Ω_η, see [21, Proposition 3]. Now, consider the composition $J_T : M \to \mathfrak{t}^*$ of the momentum map J with the restriction map from $\mathfrak{g}^*$ to $\mathfrak{t}^*$, which yields a momentum map for the

T-action on M. Since T is commutative, the coadjoint action is trivial, so that $T = T_\eta$ for all $\eta \in \mathfrak{t}^*$. Thus, $J_T^{-1}(\eta)$ is T-invariant and $J_T^{-1}(\eta)/T_\eta \simeq J_T^{-1}(\eta)/T$. Also, for regular $\eta \in \mathfrak{t}^*$ define

$$\mathcal{K}_\eta^T : H_T^*(M) \xrightarrow{\iota^*_{\eta,T}} H_T^*(J_T^{-1}(\eta)) \xrightarrow{(\pi^*_{\eta,T})^{-1}} H^*(J_T^{-1}(\eta)/T), \tag{4.18}$$

$\iota_{\eta,T} : J_T^{-1}(\eta) \hookrightarrow M$ being the inclusion and $\pi_{\eta,T} : J_T^{-1}(\eta) \to J_T^{-1}(\eta)/T$ the canonical projection. In what follows, we shall also write $\Omega_\eta^T := J_T^{-1}(\eta)$. We then have the following

Proposition 4.8. *Let $\eta \in \mathfrak{t}^*$ and $\Gamma_\eta \subset \mathfrak{t}^*$ be a conic neighborhood of the segment $\{t\eta : 0 < t < 1\}$ such that all $U_F^{\Phi^2}$ are smooth on Γ_η, and denote by $\mathfrak{t}^*_{reg}$ the set of regular values of J_T. Then, if $\varrho \in H_G^*(M)$ is an equivariantly closed form of compact support,*

$$\sum_{F\in\mathcal{F}} \mathrm{Res}^{\eta,\Lambda}(u_F\Phi^2) = (2\pi)^{d_T}\mathrm{vol}\, T \lim_{\tilde\eta\to 0,\, \tilde\eta\,\in\,\Gamma_\eta\cap\mathfrak{t}^*_{reg}} \frac{1}{|H_{\mathrm{reg}}^{\Omega^T_{\tilde\eta}}|}\int_{\Omega^T_{\tilde\eta}/T} \mathcal{K}^T_{\tilde\eta}(\mathfrak{L}), \tag{4.19}$$

where $\mathfrak{L}$ is explicitly given in terms of $e^{-i\omega}\varrho$, Φ, and J, and $d = \dim\mathfrak{g} = \dim\mathfrak{t} + |\Delta| = d_T + 2|\Delta_+|$.

Proof. Since $\mathfrak{t}^*_{\mathrm{reg}}$ is dense by Sard's theorem, the assertion is a direct consequence of (2.7) and (4.17). □

Remark 4.9. Note that if $0 \in \mathfrak{t}^*$ is a regular value of J_T, the implicit function theorem implies that the limit in (4.19) equals

$$\frac{1}{|H_{\mathrm{reg}}^{\Omega^T_0}|}\int_{\Omega^T_0/T} \mathcal{K}^T_0(\mathfrak{L}),$$

compare [11, Theorem 3, ii)]. In particular, one sees immediately that $\sum_{F\in\mathcal{F}}\mathrm{Res}^{\eta,\Lambda}(u_F\Phi^2)$ is independent of η. Further, if M is compact, the set of regular values of J_T is a disjoint union of open, convex polytopes, and $\int_{\Omega^T_\eta/T}\mathcal{K}^T_\eta(\mathfrak{L})$ is constant on each polytope [8].

4.2. **The model space and local normal form of the momentum map.** In order to set up the desingularization algorithm mentioned at the end of Section 3, we shall make use of the local normal form theorem for the momentum map J due to Guillemin-Sternberg [9] and Marle [17]. It gives a canonical description of J in a neighborhood of each G-orbit in Ω, essentially reducing J to the momentum map of a linear symplectic group action, and provides us with suitable coordinates for the desingularization process. To describe it, let p be a point in Ω, H the stabilizer of p, and $\mathcal{O} := G\cdot p \cong G/H$ the corresponding orbit in M. Let $V = (T_p(G\cdot p))^\omega/T_p(G\cdot p)$ be the fiber of the symplectic normal bundle of $\mathcal{O}$ at p, where the upper index ω denotes the symplectic complement. Fixing an $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$, we obtain an $\mathrm{Ad}(H)$-invariant splitting $\mathfrak{g} = \mathfrak{h}\times\mathfrak{m}$, where $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{m} \cong \mathfrak{g}/\mathfrak{h}$ the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. This induces a dual splitting $\mathfrak{g}^* = \mathfrak{h}^*\times\mathfrak{m}^*$. Let

$$\mathscr{T}_L \colon G\times\mathfrak{g}^* \to T^*G,\ (g,\eta)\mapsto (g,(dL_{g^{-1}})^*\eta) \tag{4.20}$$

be the trivialization of T^*G induced by left multiplication $L_a\colon G\to G$, $g\mapsto ag$. Translating it using $\mathscr{T}_L$, the chosen $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ determines a Riemannian metric on G and an associated Riemannian volume density dR. Because G is compact, we can assume (by averaging) that $dR = dg$, the Haar measure on G that we introduced earlier. Let $d\zeta$ be the Lebesgue measure on $\mathfrak{g}^*$ associated to the chosen $\mathrm{Ad}(G)$-invariant inner product (after identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via the inner product). Then, with respect to the trivialization $\mathscr{T}_L\colon G\times\mathfrak{g}^*\to T^*G$, the product measure $dgd\zeta$ on $G\times\mathfrak{g}^*$ corresponds to the Sasaki metric on T^*G. Now, in general, the two volume forms on a co-tangent bundle which are defined by the Sasaki metric with respect to any Riemannian metric on the base manifold and the canonical symplectic form, respectively, agree [14, page 537]. Therefore, with respect to $\mathscr{T}_L$, the product measure $dgd\zeta$ on $G\times\mathfrak{g}^*$ corresponds to the measure on T^*G defined by the canonical symplectic form. This will be useful later. Similarly, we obtain Lebesgue measures $d\xi$ on $\mathfrak{m}^*$ and $d\Xi$ on $\mathfrak{h}^*$ such that one has $d\zeta = d\xi d\Xi$ with respect to the decomposition $\mathfrak{g}^* = \mathfrak{h}^*\times\mathfrak{m}^*$.

We now introduce the so-called *model space* associated to the orbit $\mathcal{O}$

$$\mathcal{Y}_{\mathcal{O}} := G\times_H(\mathfrak{m}^*\times V).$$

$\mathcal{Y}_{\mathcal{O}}$ can be endowed with a symplectic structure and a Hamiltonian G-action by identifying it with the symplectic reduction of an H-action on $T^*G \times V$, where the latter space is endowed with the obvious product symplectic structure.

To describe this in more detail, note that the action induced by right multiplication $R_a \colon G \to G$, $g \mapsto ga^{-1}$ lifts to a Hamiltonian action R^* on T^*G, which with respect to the trivialization $\mathscr{T}_L$ is given by

$$R^*(a)(g,\eta) := (ga^{-1}, \mathrm{Ad}^*(a)\eta),$$

where $\mathrm{Ad}^*(a) = (\mathrm{Ad}_{a^{-1}})^*$. The corresponding momentum map $J_{R^*} \colon T^*G \to \mathfrak{g}^*$ is of the form $J_{R^*}(g,\eta) := -\eta$. Moreover, the action given by left multiplication also lifts to a Hamiltonian action on T^*G such that

$$L^*(a)(g,\eta) := (ag,\eta)$$

with corresponding momentum map $J_{L^*}(g,\eta) := \mathrm{Ad}^*(g)\eta$. Let $R^*_H := R^*|_H$; then R^*_H is still a Hamiltonian action on T^*G and the corresponding momentum map $\Phi_{R^*_H}$ is given by J_{R^*} followed by the orthogonal projection of $\mathfrak{g}^*$ to $\mathfrak{h}^*$.

On the other hand, consider the linear symplectic Hamiltonian action $\mathscr{H} \colon H \to Sp(V,\omega_V)$ on V with H-equivariant momentum map Φ_V given by

$$\langle X, \Phi_V(v)\rangle = \frac{1}{2}\omega_V(X_V \cdot v, v), \qquad X \in \mathfrak{h},\, v \in V, \tag{4.21}$$

where we wrote $X_V := (d\mathscr{H})(X) \in \mathfrak{sp}(V,\omega_V)$. Taking everything together, we obtain a Hamiltonian product action of H on $T^*G \times V$ with H-equivariant momentum map $\Phi \colon G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V \to \mathfrak{h}^*$ given by the formula

$$\Phi(g,\xi,\xi',v) := \Phi_V(v) - \xi',$$

which is simply the sum $\Phi_{R^*_H} + \Phi_V$. Since zero is a regular value of Φ, the reduced space $\Phi^{-1}(0)/H$ is a symplectic manifold and can be identified with our model space $\mathcal{Y}_{\mathcal{O}}$ via the following H-equivariant diffeomorphism

$$G \times \mathfrak{m}^* \times V \to \Phi^{-1}(0) \subset G \times \mathfrak{m}^* \times \mathfrak{h}^* \times V, \quad (g,\xi,v) \mapsto (g,\xi,\Phi_V(v),v). \tag{4.22}$$

Hence $\mathcal{Y}_{\mathcal{O}} \cong \Phi^{-1}(0)/H$ inherits a symplectic structure from $T^*G \times V$, and one can show that the embedding of G/H into $\mathcal{Y}_{\mathcal{O}}$ is isotropic with symplectic normal bundle $G \times_H V$.

Next, let us describe the Hamiltonian G-action on the model space. To do this, we first define an action $\mathscr{L}$ on $T^*G \times V$ by letting G act on T^*G by L^* and trivially on V. The momentum map $\mathscr{J}$ associated to $\mathscr{L}$ is then simply given by $\mathscr{J}(g,\eta,v) = \mathrm{Ad}^*(g)\eta$ with respect to the trivialization $T^*G \times V \simeq G \times \mathfrak{g}^* \times V$. Since $\mathscr{L}$ commutes with the action of H on $T^*G \times V$ it descends to an action $\mathcal{L}_{\mathcal{O}}$ on $\mathcal{Y}_{\mathcal{O}}$. Thus, if $\pi : G \times \mathfrak{m}^* \times V \to \big(G \times \mathfrak{m}^* \times V\big)/H = \mathcal{Y}_{\mathcal{O}}$ denotes the canonical projection, the square

$$\begin{tikzcd}
G \times \mathfrak{m}^* \times V \arrow[r, "\mathscr{L}(g)"] \arrow[d, "\pi = /H"] & G \times \mathfrak{m}^* \times V \arrow[d, "\pi = /H"] \\
\mathcal{Y}_{\mathcal{O}} \arrow[r, "\mathcal{L}_{\mathcal{O}}(g)"] & \mathcal{Y}_{\mathcal{O}}
\end{tikzcd} \tag{4.23}$$

commutes for each $g \in G$, and this is the defining property of $\mathcal{L}_{\mathcal{O}}$. Moreover, since the momentum map J_{L^*} is H-invariant, the momentum map $\mathcal{J}_{\mathcal{O}}$ of the G-action $\mathcal{L}_{\mathcal{O}}$ is given by

$$\mathcal{J}_{\mathcal{O}} \colon G \times_H (\mathfrak{m}^* \times V) \to \mathfrak{g}^*, \quad [g,\xi,v] \mapsto \mathscr{J}_{|\Phi^{-1}(0)}(g,\xi + \Phi_V(v), v) = \mathrm{Ad}^*(g)(\xi + \Phi_V(v)), \tag{4.24}$$

where $[g,\xi,v]$ is the image of (g,ξ,v) under the quotient map $G \times \mathfrak{m}^* \times V \to G \times_H (\mathfrak{m}^* \times V)$. In other words, the triangle

(4.25)

$$
\begin{tikzcd}
\Phi^{-1}(0) \simeq G \times \mathfrak{m}^* \times V \arrow[d, "\pi"] \arrow[r, "\mathscr{J}_{|\Phi^{-1}(0)}"] & \mathfrak{g}^* \\
\mathcal{Y}_{\mathcal{O}} \arrow[ur, "\mathcal{J}_{\mathcal{O}}"'] &
\end{tikzcd}
$$

commutes. But note that $\mathscr{J}_{|\Phi^{-1}(0)}$ is not a momentum map. As a consequence of the isotropic embedding theorem we now obtain the following very useful relation between the model space $\mathcal{Y}_{\mathcal{O}}$ and our symplectic G-manifold M.

Theorem 4.10 (Local normal form for the momentum map, [22, Prop. 2.5])**.** *For each G-orbit $\mathcal{O} \cong G/H$ in Ω there is an open neighbourhood $U_{\mathcal{O}}$ of $\mathcal{O}$ in M and a G-equivariant symplectomorphism $\varphi_{\mathcal{O}} : U_{\mathcal{O}} \xrightarrow{\cong} \mathcal{U}_{\mathcal{O}} \subset \mathcal{Y}_{\mathcal{O}}$ onto an open neighbourhood of the zero section in $\mathcal{Y}_{\mathcal{O}}$ such that the momentum map has the* local normal form (4.24)*.*

□

Now, for some chosen G-orbit $\mathcal{O} \subset \Omega$, put $\mathscr{U}_{\mathcal{O}} := \pi^{-1}(\mathcal{U}_{\mathcal{O}})$. Combining the statement of Theorem 4.10 with the commutative triangle (4.25), we obtain the commutative diagram

$$
\begin{tikzcd}
\Phi^{-1}(0) \simeq G \times \mathfrak{m}^* \times V \arrow[d, "\pi = /H"] & \mathscr{U}_{\mathcal{O}} \arrow[l, "\supset"'] \arrow[d, "\pi = /H"] \arrow[dr, "\mathscr{J}_{|\Phi^{-1}(0)}"] & \\
\mathcal{Y}_{\mathcal{O}} & \mathcal{U}_{\mathcal{O}} \arrow[l, "\supset"'] \arrow[r, "\mathcal{J}_{\mathcal{O}}"] \arrow[d, "\varphi_{\mathcal{O}}^{-1}"] & \mathfrak{g}^* \\
& U_{\mathcal{O}} \arrow[ul, "\varphi_{\mathcal{O}}"] \arrow[ur, "J|_{U_{\mathcal{O}}}"] \arrow[d, "\subset"'] & \\
& M \arrow[uur, "J"'] &
\end{tikzcd}
\tag{4.26}
$$

Strictly spoken, the maps $\mathscr{J}_{|\Phi^{-1}(0)}$ and $\mathcal{J}_{\mathcal{O}}$ in the diagram are the *restrictions* to the open sets $\mathscr{U}_{\mathcal{O}}$ and $\mathcal{U}_{\mathcal{O}}$, respectively, but we omitted the restriction symbols for a clearer notation. Now, the Marle-Guillemin-Sternberg construction gives us an easy local description of the stratum in Ω corresponding to the type (H) of the orbit $\mathcal{O}$. Namely, by [22, p. 386], one has

$$
\mathcal{Y}_{\mathcal{O}}(H) \cap \mathcal{J}_{\mathcal{O}}^{-1}(0) = \{[g, \xi, v] \in \mathcal{Y}_{\mathcal{O}} : \xi = 0, v \in V_H\}, \tag{4.27}
$$

where $V_H \subset V$ is the linear subspace of those vectors in V that are fixed by the H-action. V_H is a symplectic subspace of V. Thus, the image in $\mathcal{U}_{\mathcal{O}}$ of $\Omega_{(H)}$ is simply given by

$$
\varphi_{\mathcal{O}}\big(\Omega_{(H)} \cap U_{\mathcal{O}}\big) = \{[g, \xi, v] \in \mathcal{U}_{\mathcal{O}} : \xi = 0, v \in V_H\} = G/H \times \mathscr{V}_H, \tag{4.28}
$$

where $\mathscr{V}_H \subset V_H$ is an open neighbourhood of the origin, and consequently $\varphi_{\mathcal{O}}$ induces a symplectomorphism

$$
\widetilde{\varphi}_{\mathcal{O}} : (\Omega_{(H)} \cap U_{\mathcal{O}})/G \xrightarrow{\cong} \mathscr{V}_H \tag{4.29}
$$

which maps $\mathcal{O}$ to 0. In particular, $\widetilde{\varphi}_{\mathcal{O}}$ identifies the restriction of the natural symplectic volume form on V_H to $\mathscr{V}_H$ with the restriction of the natural symplectic volume form on $\Omega_{(H)}/G$ to $(\Omega_{(H)} \cap U_{\mathcal{O}})/G$.

Let us mention the following direct corollary of Theorem 4.10:

Corollary 4.11. *The dimension of each orbit $\mathcal{O} \subset \Omega$, $\mathcal{O} \cong G/H$, satisfies the inequality*

$$
\dim \mathcal{O} \leq \frac{1}{2}(\dim M - \dim \Omega_{(H)}/G).
$$

Proof. By Theorem 4.10 and the construction of the model space $\mathcal{Y}_{\mathcal{O}}$, we have

$$\dim M = \dim \mathcal{Y}_{\mathcal{O}} = \dim G + \dim \mathfrak{m}^* + \dim V_H + \dim W - \dim H,$$

where W is the symplectic complement of V_H in V. Now, by definition, one also has

$$\dim \mathfrak{m}^* = \dim G - \dim H = \dim \mathcal{O}, \tag{4.30}$$

and (4.29) implies that $\dim V_H = \dim \Omega_{(H)}/G$. Thus, we arrive at

$$0 \leq \dim W = \dim M - \dim \Omega_{(H)}/G - 2\dim \mathcal{O}. \tag{4.31}$$

□

4.3. **The reduced model phase function and its critical set.** Recall from (4.2) that the critical set of our phase function $\psi : M \times \mathfrak{g} \to \mathbb{R}$ reads

$$\mathrm{Crit}(\psi) = \{(p, X) \in M \times \mathfrak{g} : \psi_*(p, X) = 0\} = \big\{(p, X) \in \Omega \times \mathfrak{g} : \widetilde{X}_p = 0\big\},$$

and that our purpose is to produce a clean monomialization for ψ or, equivalently, for $\mathcal{J}_{\mathcal{O}}([g, \xi, v])(X)$ in $\mathcal{Y}_{\mathcal{O}} \times \mathfrak{g}$. To this end, it will actually be convenient to consider the *model phase function*

$$\psi_{\mathcal{O}} : G \times \mathfrak{m}^* \times V \times \mathfrak{g} \longrightarrow \mathbb{R}, \quad (g, \xi, v, X) \longmapsto \mathscr{J}_{|\Phi^{-1}(0)}(g, \xi, v)(X) \equiv \mathrm{Ad}^*(g)(\xi + \Phi_V(v))(X), \tag{4.32}$$

which will be easier to handle than working with $\mathcal{J}_{\mathcal{O}}$ in the quotient space. Let $(g, \xi, v, X) \in G \times \mathfrak{m}^* \times V \times \mathfrak{g}$ be a point in the critical set of $\psi_{\mathcal{O}}$. Setting the partial derivatives of $\psi_{\mathcal{O}}$ with respect to the X-variables to 0 yields the necessary condition

$$\xi = 0, \qquad v \in \Phi_V^{-1}(0), \tag{4.33}$$

while the vanishing of the ξ-derivatives of $\psi_{\mathcal{O}}$ yields $\mathrm{pr}_{\mathfrak{m}}\big(\mathrm{Ad}(g)(X)\big) = 0$. The evaluation of the v-derivatives of $\psi_{\mathcal{O}}$ gives $(\mathrm{pr}_{\mathfrak{h}} \circ \mathrm{Ad}(g)(X))_V \cdot v = 0$, which together with the previous condition is equivalent to

$$X \in \mathrm{Ad}(g)^{-1}(\mathfrak{h}_v), \tag{4.34}$$

where $\mathfrak{h}_v \subset \mathfrak{h}$ is the isotropy algebra of the point v. Finally, the vanishing of the g-derivatives of $\psi_{\mathcal{O}}$ yields a condition which is vacuous once (4.33) is fulfilled. On the other hand, consider the set of (g, ξ, v, X) on which (4.34) is fulfilled. Then ψ_O is constantly zero on this set and hence its derivative vanishes. In summary, we have found that

$$\begin{aligned} \mathrm{Crit}(\psi_{\mathcal{O}}) &= \big\{(g, \xi, v, X) \in G \times \mathfrak{m}^* \times V \times \mathfrak{g} : \xi = 0,\ v \in \Phi_V^{-1}(0),\ X \in \mathrm{Ad}(g)^{-1}(\mathfrak{h}_v)\big\} \\ &= \big\{(g, \xi, v, X) \in G \times \mathfrak{m}^* \times V \times \mathfrak{g} : [g, \xi, v] \in \mathcal{J}_{\mathcal{O}}^{-1}(0),\ (\mathrm{Ad}\,(g)X)_V \cdot v = 0\big\}, \end{aligned} \tag{4.35}$$

in accordance with (4.2). The origins of possible singularities are clearly visible; they can arise from singularities of $\Phi_V^{-1}(0)$ and jumping dimensions of the isotropy algebra $\mathfrak{h}_v$ as v varies.

Next, consider the decomposition

$$V = V_H \oplus W, \tag{4.36}$$

where W is the symplectic complement of V_H in V. By definition, W is a symplectic vector space with the symplectic structure inherited from V, and the symplectic H-action on V restricts to a symplectic H-action on W, with the corresponding momentum map Φ_W being given by the restriction of Φ_V to W. Since V_H is precisely the subspace in V of those vectors fixed by the H-action, it is easy to check that when we decompose a vector $v \in V$ into $v = v' + w$ with $v' \in V_H$, $w \in W$, one has

$$\Phi_V(v) = \Phi_W(w), \tag{4.37}$$

which means that Φ_V is constant in the directions of V_H. It follows that

$$\Phi_V^{-1}(0) = V_H \times \Phi_W^{-1}(0), \tag{4.38}$$

see [22, Eq. (4)]. Thus, for our purposes it will be sufficient to deal with the *reduced model phase function*

$$\widetilde{\psi}_{\mathcal{O}} : G \times \mathfrak{m}^* \times W \times \mathfrak{g} \longrightarrow \mathbb{R}, \quad (g, \xi, w, X) \longmapsto \mathrm{Ad}^*(g)(\xi + \Phi_W(w))(X), \tag{4.39}$$

and analogous computations as before yield

$$\mathrm{Crit}(\widetilde{\psi}_{\mathcal{O}}) \; = \; \big\{(g,\xi,w,X) \in G \times \mathfrak{m}^* \times W \times \mathfrak{g} : \xi = 0, \; w \in \Phi_W^{-1}(0), \; X \in \mathrm{Ad}(g)^{-1}(\mathfrak{h}_w)\big\}.$$

Now, $\widetilde{\psi}_{\mathcal{O}}$ splits into a sum

$$\begin{aligned} \widetilde{\psi}_{\mathcal{O}}(g,\xi,w,X) &= \xi(\mathrm{pr}_{\mathfrak{m}} \circ \mathrm{Ad}\,(g)X) + \Phi_W(w)(\mathrm{pr}_{\mathfrak{h}} \circ \mathrm{Ad}\,(g)X) \\ &=: \psi_{\mathfrak{m}^*}(\xi, \mathrm{pr}_{\mathfrak{m}} \circ \mathrm{Ad}\,(g)X) + \psi_W(w, \mathrm{pr}_{\mathfrak{h}} \circ \mathrm{Ad}\,(g)X), \end{aligned} \tag{4.40}$$

allowing us to treat the term linear in ξ and the one quadratic in w separately. While $\psi_{\mathfrak{m}^*} : \mathfrak{m}\times\mathfrak{m}^* \to \mathbb{R}$ is not a momentum map, $\psi_W : W \times \mathfrak{h} \to \mathbb{R}$ is given in terms of the standard momentum map of a linear symplectic group action on linear symplectic vector space treated in Section 5.3. In particular, discarding the G-dependence, the relevant critical sets read

$$\begin{aligned} \mathrm{Crit}(\psi_{\mathfrak{m}^*}) \; &= \; \big\{(\xi, X) \in \mathfrak{m}^* \times \mathfrak{m} : \xi = 0, \; X = 0\big\}, \\ \mathrm{Crit}(\psi_W) \; &= \; \big\{(w, X) \in W \times \mathfrak{h} : \; w \in \Phi_W^{-1}(0), \; X \in \mathfrak{h}_w\big\}, \end{aligned}$$

$\mathrm{Crit}(\psi_{\mathfrak{m}^*})$ being clean, while $\mathrm{Crit}(\psi_W)$ is clearly singular if there are orbits of at least two different dimensions in $\Phi_W^{-1}(0)$.

4.4. **Isotropy types and symplectic slices.** With the notation introduced before, we shall now collect some basic relations between the G-isotropy types occurring in $U_{\mathcal{O}}$ and the H-isotropy types occurring in V and W.

Proposition 4.12. *Only G-isotropy types (H') with $(H') \geq (H)$ occur in $U_{\mathcal{O}}$. The set of H-isotropy types in V corresponds to a subset of the set of G-isotropy types occurring in $U_{\mathcal{O}}$. The H-isotropy types in $\Phi_V^{-1}(0)$ correspond one-to-one to the G-isotropy types occurring in $U_{\mathcal{O}} \cap \Omega$.*

Proof. Let $p \in U_{\mathcal{O}}$, $p = \varphi_{\mathcal{O}}^{-1}([g_0, \xi_0, v_0])$, and let $g \in G_p$, so that

$$g \cdot \varphi_{\mathcal{O}}^{-1}([g_0, \xi_0, v_0]) = \varphi_{\mathcal{O}}^{-1}([g_0, \xi_0, v_0]). \tag{4.41}$$

Since $g \cdot \varphi_{\mathcal{O}}^{-1}([g_0, \xi_0, v_0]) = \varphi_{\mathcal{O}}^{-1}([gg_0, (\partial g)^T \xi_0, v_0])$, and φ^{-1} is bijective, (4.41) is equivalent to

$$[gg_0, (\partial g)^T \xi_0, v_0] = [g_0, \xi_0, v_0], \tag{4.42}$$

which in turn is equivalent to the statement that there is an element $h \in H$ such that

$$(gg_0, (\partial g)^T \xi_0, v_0) = (g_0 h, (\partial(\cdot h))^T \xi_0, v_0 \cdot h).$$

This implies that $h \in H_{v_0}$ and $g = g_0 h g_0^{-1}$, so that

$$G_p = G_{\varphi_{\mathcal{O}}^{-1}([g_0,\xi_0,v_0])} = g_0 H' g_0^{-1}, \tag{4.43}$$

where H' is the subgroup of H_{v_0} given by those $h \in H_{v_0}$ that fulfill $(\partial g_0^{-1})^T (\partial h)^T (\partial g_0)^T \xi_0 = (\partial(\cdot h))^T \xi_0$. In particular, for an arbitrary $v \in V$, we can choose p such that $v_0 = v$, $\xi_0 = 0$, and $g_0 = e$, and then we get

$$H_v = G_{\varphi_{\mathcal{O}}^{-1}([e,0,v])}. \tag{4.44}$$

This shows that all H-isotropy types in V occur as G-isotropy types in $U_{\mathcal{O}}$, which proves the second statement of the proposition. On the other hand, (4.43) implies that every G-isotropy group occurring in $U_{\mathcal{O}}$ is conjugate in G to a subgroup of an H-isotropy group in V, which proves the first statement. Finally, if $[(g, \xi, v)]$ lies in the zero level of the momentum map $\mathcal{J}_{\mathcal{O}}$, then one has $\xi = 0$, and (4.43) yields

$$G_{\varphi_{\mathcal{O}}^{-1}([g,0,v])} = g_0 H_v g_0^{-1} \qquad \forall \, [g, 0, v] \in \mathcal{J}_{\mathcal{O}}^{-1}(0), \tag{4.45}$$

which implies that

$$\Phi_V^{-1}(0)(H') = (\mathfrak{m}^* \times V) \cap \mathcal{J}_{\mathcal{O}}^{-1}(0) \cap \mathcal{Y}_{\mathcal{O}}(H'), \tag{4.46}$$

for any closed subgroup $H' \subset H$, where $(\mathfrak{m}^* \times V)$ denotes the typical fiber of the bundle $\mathfrak{m}^* \times V \to \mathcal{Y}_{\mathcal{O}} \to G/H$, and the equality should be understood up to trivial identifications like $\{0\} \times V = V$. Now,

since the H-action on V is linear, the isotropy types occurring in the intersection of $\Phi_V^{-1}(0)$ with any open neighbourhood of the origin in V are the same as those occurring in all of $\Phi_V^{-1}(0)$. Therefore, by (4.46), the set $\Phi_V^{-1}(0)(H')$ is non-empty iff $\mathcal{J}_{\mathcal{O}}^{-1}(0) \cap \mathcal{U}_{\mathcal{O}}(H')$ is non-empty. Since $\mathcal{U}_{\mathcal{O}}$ is equivariantly symplectomorphic to $U_{\mathcal{O}}$, we conclude that $\Phi_V^{-1}(0)(H')$ is non-empty iff $\Omega \cap U_{\mathcal{O}}(H')$ is non-empty, and this holds for any closed subgroup $H' \subset H$. □

An immediate consequence is the following

Corollary 4.13. *The set of H-isotropy types in $W - \{0\}$ corresponds to a subset of the set of G-isotropy types occurring in $U_{\mathcal{O}}$ which are strictly greater than (H). The H-isotropy types occurring in $\Phi_W^{-1}(0) - \{0\}$ correspond one-to-one to the G-isotropy types occurring in $U_{\mathcal{O}} \cap \Omega$ except (H). In particular, if (H) is the only G-isotropy type occurring in $U_{\mathcal{O}}$, then $W = \{0\}$, and if (H) is the only G-isotropy type occurring in $U_{\mathcal{O}} \cap \Omega$, then $\Phi_W^{-1}(0) = \{0\}$.*

Proof. By construction of $W \subset V$, we have $\{0\} = W \cap V_H = W \cap V(H)$. This means that the isotropy types of the H-action in $W - \{0\}$ are precisely those of the H-action in V except (H). The statements now follow directly from the previous proposition. □

The following observations will be useful later.

Proposition 4.14. *The zero level set $\Phi_W^{-1}(0)$ of the momentum map in W is connected. The regular G-isotropy types in all connected components of $U_{\mathcal{O}} \cap \Omega$ agree and correspond to the regular H-isotropy type in $\Phi_W^{-1}(0)$.*

Proof. Since $\Phi_W(w)$ is quadratic in w, one has that $\lambda w \in \Phi_W^{-1}(0)$ for every $w \in \Phi_W^{-1}(0)$ and any $\lambda \in \mathbb{R}$. Thus, $\Phi_W^{-1}(0)$ is path-connected because every point w in it is connected to $0 \in \Phi_W^{-1}(0)$ via the straight line $[0, w]$. To prove the second statement, we recall that the characteristic property of the regular isotropy type (H_{reg}) in $\Phi_W^{-1}(0)$ is that the associated stratum $\Phi_W^{-1}(0)_{(H_{\mathrm{reg}})}$ is open and dense in $\Phi_W^{-1}(0)$. By (4.46) and (4.38), one has

$$V_H \times \Phi_W^{-1}(0)_{(H_{\mathrm{reg}})} = \Phi_V^{-1}(0)_{(H_{\mathrm{reg}})} = (\mathfrak{m}^* \times V) \cap \mathcal{J}_{\mathcal{O}}^{-1}(0) \cap \mathcal{Y}_{\mathcal{O}(H_{\mathrm{reg}})}, \tag{4.47}$$

where $(\mathfrak{m}^* \times V)$ denotes the typical fiber of the bundle $\mathfrak{m}^* \times V \to \mathcal{Y}_{\mathcal{O}} \to G/H$, and the equality should be understood up to trivial identifications like $\{0\} \times V \equiv V$. Similarly, one has the relation

$$V_H \times \Phi_W^{-1}(0) = \Phi_V^{-1}(0) = (\mathfrak{m}^* \times V) \cap \mathcal{J}_{\mathcal{O}}^{-1}(0). \tag{4.48}$$

We see that $V_H \times \Phi_W^{-1}(0)_{(H_{\mathrm{reg}})}$ is the typical fiber of the bundle

$$\Phi_V^{-1}(0)_{(H_{\mathrm{reg}})} \to \mathcal{J}_{\mathcal{O}}^{-1}(0)_{(H_{\mathrm{reg}})} \to G/H,$$

while $V_H \times \Phi_W^{-1}(0)$ is the typical fiber of the bundle $\Phi_V^{-1}(0) \to \mathcal{J}_{\mathcal{O}}^{-1}(0) \to G/H$ of topological spaces. This means that locally one has homeomorphisms

$$\mathcal{J}_{\mathcal{O}}^{-1}(0) \cong G/H \times V_H \times \Phi_W^{-1}(0), \qquad \mathcal{J}_{\mathcal{O}}^{-1}(0)_{(H_{\mathrm{reg}})} \cong G/H \times V_H \times \Phi_W^{-1}(0)_{(H_{\mathrm{reg}})}. \tag{4.49}$$

We conclude that the connected components of $\mathcal{J}_{\mathcal{O}}^{-1}(0)$ correspond to the connected components of G/H, and since G acts transitively on G/H, the regular isotropy types in all connected components of $\mathcal{J}_{\mathcal{O}}^{-1}(0)$ agree. It follows directly from the existence of the local homeomorphisms (4.49) that the intersection of $\mathcal{J}_{\mathcal{O}}^{-1}(0)_{(H_{\mathrm{reg}})}$ with some connected component of $\mathcal{J}_{\mathcal{O}}^{-1}(0)$ is open and dense in that component if, and only if, $\Phi_W^{-1}(0)_{(H_{\mathrm{reg}})}$ is open and dense in $\Phi_W^{-1}(0)$. This proves that the regular isotropy type in $\Phi_W^{-1}(0)$, considered as a G-isotropy type, agrees with the regular isotropy types in all connected components of $\mathcal{J}_{\mathcal{O}}^{-1}(0)$. Finally, by Theorem 4.10, an open neighbourhood of the zero section in $\mathcal{J}_{\mathcal{O}}^{-1}(0)$ is equivariantly homeomorphic to $U_{\mathcal{O}} \cap \Omega$, so that the regular isotropy types in all connected components of $U_{\mathcal{O}} \cap \Omega$ agree with the regular isotropy types in the corresponding subsets of $\mathcal{J}_{\mathcal{O}}^{-1}(0)$. □

5. The desingularization process

We shall now implement an iterative desingularization procedure that will allow us to describe the leading term in the asymptotic expansion of the integral (2.9) as $\mu \to 0^+$ together with a remainder estimate in case that $\eta = 0$ is not necessarily a regular value of the momentum map. Each iteration step of the desingularization will consist of

- a decomposition by orbit types,
- a reduction to the linear symplectic case by means of the local normal form theorem for the momentum map,
- an evaluation of oscillatory integrals of the simple form
$$\int_{\mathbb{R}^{2n}} e^{i\langle x,\xi\rangle/\nu} a(x,\xi)\, dx\, d\xi, \qquad a \in \mathrm{C}^\infty_\mathrm{c}(\mathbb{R}^{2n}),\ \nu \to 0^+,$$
- a discretized blow-up.

The desingularization algorithm either stops if the critical set of the phase function is already clean, or otherwise produces an output that is formally identical to its input, but less singular, so that one can *repeat* the whole procedure until it stops naturally.

5.1. **Decomposition using symplectic slices.** We begin by introducing a suitable covering of the manifold M that will allow us to make use of the local normal form of the momentum map described in Theorem 4.10. With the notation as before, we fix a G-orbit $\mathcal{O} \subset \Omega$ and a closed subgroup H of G such that $\mathcal{O} \cong G/H$. Let $U_\mathcal{O} \subset M$ be an open neighbourhood of $\mathcal{O}$ as in Theorem 4.10. In what follows, we will use the more explicit notations

$$G_\mathcal{O} := H, \quad \mathfrak{g}_\mathcal{O} := \mathfrak{h}, \quad V_\mathcal{O} := V, \quad W_\mathcal{O} := W, \quad \mathfrak{m}_\mathcal{O} := \mathfrak{m}, \quad \Omega_M := \Omega, \quad J_M := J, \quad \psi_M := \psi.$$

We will now describe a few technical assumptions about the sets $U_\mathcal{O}, \mathcal{U}_\mathcal{O}$, and $\mathscr{U}_\mathcal{O}$ that are not a priori fulfilled, but which we can assume without loss of generality so that the assertions of Theorem 4.10 still hold. These assumptions will simplify the computations later.

- By replacing $U_\mathcal{O}$ with a smaller open set and taking into account that G and $G_\mathcal{O}$ are compact, we can (and will) assume that the sets $U_\mathcal{O}, \mathcal{U}_\mathcal{O}$, and $\mathscr{U}_\mathcal{O} = \pi^{-1}(\mathcal{U}_\mathcal{O})$ are pre-compact. By making $U_\mathcal{O}$ even smaller, we can assume that the diffeomorphism $\varphi_\mathcal{O}$ extends to a compact set containing $U_\mathcal{O}$, which implies that the supremum norm of each partial derivative of $\varphi_\mathcal{O}$ is finite on $U_\mathcal{O}$.
- Since $G_\mathcal{O}$ is compact, we can equip $W_\mathcal{O}$ with a $G_\mathcal{O}$-invariant inner product and define
$$\mathscr{B}_\mathcal{O}(r) := \{w \in W_\mathcal{O} : \|w\| < r\}, \qquad r > 0,$$
the associated open ball of radius r. By construction, the latter is $G_\mathcal{O}$-invariant, so that the symplectic $G_\mathcal{O}$-action descends to an $G_\mathcal{O}$-action on $\mathscr{B}_\mathcal{O}(r)$. By passing to an even smaller G-invariant open neighourhood of $\mathcal{O}$ and rescaling the $G_\mathcal{O}$-invariant inner product on $W_\mathcal{O}$ we can assume that $\mathscr{U}_\mathcal{O} = \pi^{-1}(\mathcal{U}_\mathcal{O})$ has the form
$$\mathscr{U}_\mathcal{O} = G \times \mathscr{M}_\mathcal{O} \times \mathscr{V}_\mathcal{O} \times \mathscr{B}_\mathcal{O}(1/2), \tag{5.1}$$
where $\mathscr{M}_\mathcal{O} \subset \mathfrak{m}^*_\mathcal{O}$ and $\mathscr{V}_\mathcal{O} \subset V_{\mathcal{O},G_\mathcal{O}}$ are connected open pre-compact neighbourhoods of the origin, respectively, such that $\mathscr{M}_\mathcal{O}$ is G-invariant.

Assigning in this way to each orbit $\mathcal{O}$ in Ω_M the set $U_\mathcal{O}$ that fulfills the assumptions above we obtain an open cover $\{U_\mathcal{O} : \mathcal{O} \subset \Omega_M\}$ of Ω_M in which each open set corresponds to a G-invariant neighbourhood $\mathcal{U}_\mathcal{O}$ of the zero section in the associated model space, and, by Proposition 4.12, is free from any lower orbit types. Since M (and hence Ω_M) is second countable and the support of the amplitude a is compact, we can choose a countable set $\aleph(M)$ of orbits such that

$$\mathbf{U}_M := \{U_\mathcal{O} : \mathcal{O} \in \aleph(M)\}$$

is still an open cover of Ω_M and only finitely many open sets have non-empty intersection with supp a. In general, all orbit types occurring in Ω_M may occur as orbit types associated to orbits in $\aleph(M)$. Note that as J_M is continuous, $\Omega_M = J_M^{-1}(0)$ is closed in M, hence $\mathbf{U}_M \cup \{M - \Omega_M\}$ is an open cover

of M. Let $\{\chi_{U_\mathcal{O}}\}_{\mathcal{O}\in\aleph} \cup \{\chi_{M-\Omega_M}\}$ be a partition of unity subordinate to this cover of M such that each function $\chi_{U_\mathcal{O}}$ is G-invariant. With the notation of (2.9) define

$$I^{\square}(\mu) := \int\limits_{\mathfrak{g}} \int\limits_{M-\Omega_M} e^{i\psi_M(p,X)/\mu} a(p,X)\chi_{M-\Omega_M}(p)\, dp\, dX, \tag{5.2}$$

$$I_\mathcal{O}(\mu) := \int\limits_{\mathfrak{g}} \int\limits_{U_\mathcal{O}} e^{i\psi_M(p,X)/\mu} a(p,X)\chi_{U_\mathcal{O}}(p)\, dp\, dX, \qquad \mathcal{O} \in \aleph(M), \tag{5.3}$$

so that

$$I(\mu) = \sum_{\mathcal{O}\in\aleph(M)} I_\mathcal{O}(\mu) + I^{\square}(\mu) = \int\limits_{\mathfrak{g}} \int\limits_{M} e^{i\psi_M(p,X)/\mu} a(p,X)\, dp\, dX. \tag{5.4}$$

Note that each cutoff function $\chi_{U_\mathcal{O}}$ is necessarily compactly supported since we assume $U_\mathcal{O}$ to be pre-compact, so that in each of the integrals $I_\mathcal{O}(\mu)$ the support of the integrand is contained in a compact set that is independent of the amplitude a. In contrast, $\chi_{M-\Omega_M}$ is not necessarily compactly supported unless M is compact, and therefore the integrand in $I^{\square}(\mu)$ is only compactly supported because a is. We proceed with the observation that, by (4.2), ψ_M has no critical points in supp $\chi_{M-\Omega_M} \times \mathfrak{g}$, and consequently

$$I^{\square}(\mu) = \mathrm{O}\big(\mu^k \cdot \big\|\Delta_{\mathfrak{g}}^{k/2} a\big\|_\infty \cdot \text{vol supp } a\big) \qquad \text{as } \mu \to 0^+ \qquad \forall\, k > 0, \tag{5.5}$$

where $\Delta_\mathfrak{g}$ is the Laplacian on $\mathfrak{g}$ with respect to our chosen inner product, or more precisely the pullback of this Laplacian to $M \times \mathfrak{g}$. The implicit constants in the estimate do not depend on a but of course on the choice of the cutoff function $\chi_{M-\Omega_M}$. To derive an asymptotic formula for $I(\mu)$ as $\mu \to 0^+$, it suffices to consider for each individual $\mathcal{O} \in \aleph(M)$ the integral $I_\mathcal{O}(\mu)$. In what follows, we shall reduce this problem to the linear symplectic case by means of the model space $\mathcal{Y}_\mathcal{O}$.

5.2. **Reduction to the linear symplectic case.** For the rest of this sub-section, let us fix an orbit $\mathcal{O} \in \aleph(M)$ and a closed subgroup $G_\mathcal{O} \subset G$ representing the isotropy type of $\mathcal{O}$. We begin by lifting the integral $I_\mathcal{O}(\mu)$ to the open set $\mathcal{U}_\mathcal{O} \subset \mathcal{Y}_\mathcal{O}$ in the model space along the G-equivariant symplectomorphism $\varphi_\mathcal{O} : U_\mathcal{O} \xrightarrow{\cong} \mathcal{U}_\mathcal{O}$. This yields for $I_\mathcal{O}(\mu)$ the expression

$$\int\limits_{\mathfrak{g}} \int\limits_{U_\mathcal{O}} e^{i\psi_M(p,X)/\mu} a(p,X)\chi_{U_\mathcal{O}}(p)\, dp\, dX = \int\limits_{\mathfrak{g}} \int\limits_{\mathcal{U}_\mathcal{O}} e^{i\psi_M(\varphi_\mathcal{O}^{-1}(y),X)/\mu} a\big(\varphi_\mathcal{O}^{-1}(y), X\big)\chi_{U_\mathcal{O}}(\varphi_\mathcal{O}^{-1}(y))\, dy\, dX,$$

where dy is the symplectic volume form in $\mathcal{Y}_\mathcal{O}$, which is precisely the pullback of the symplectic volume form dp on M under $\varphi_\mathcal{O}^{-1}$. Since our model phase function $\psi_\mathcal{O}$ is defined on $G \times \mathfrak{m}_\mathcal{O}^* \times V_\mathcal{O}$, see (4.32), we want to lift $I_\mathcal{O}(\mu)$ still one level higher in the diagram (4.26) to the set $\mathscr{U}_\mathcal{O} \subset G \times \mathfrak{m}_\mathcal{O}^* \times V_\mathcal{O}$. On $G \times \mathfrak{m}_\mathcal{O}^* \times V_\mathcal{O}$, we have no canonical volume form due to the lack of a symplectic structure. But since $\pi\colon G \times \mathfrak{m}_\mathcal{O}^* \times V_\mathcal{O} \to \mathcal{Y}_\mathcal{O}$ is a principal $G_\mathcal{O}$-bundle there exists for every volume density du on $G \times \mathfrak{m}_\mathcal{O}^* \times V_\mathcal{O}$ a form η on the same space such that $du = |\pi^*(dy) \wedge \eta|$ and the restriction of η to each fiber of π defines a volume density denoted by η_y for $y \in \mathcal{Y}_\mathcal{O}$ (cf [16, p. 430]). Then by [16, Theorem 4.8], we have for any continuous function f with compact support on $\mathcal{U}_\mathcal{O}$ the equality

$$\int\limits_{\mathscr{U}_\mathcal{O}} \pi^*(f)\, du = \int\limits_{\mathcal{U}_\mathcal{O}} \int\limits_{\pi^{-1}\{y\}} \pi^*(f)\eta_y\, dy = \int\limits_{\mathcal{U}_\mathcal{O}} \pi^*(f) \cdot \beta(y)\, dy, \tag{5.6}$$

where $\pi^*(f)$ denotes the pullback of f by π and $\beta(y) = \int_{\pi^{-1}\{y\}} \eta_y$ the volume of the fiber over y with respect to η_y. Note that $\pi^*(f)$ has still compact support on $\mathscr{U}_\mathcal{O}$ since the fibers are compact. Applying this to our integral $I_\mathcal{O}$ and taking into account the defining relation $\psi_\mathcal{O} \equiv \psi_M \circ (\varphi_\mathcal{O}^{-1} \circ \pi \times \mathrm{id}_{\,\mathfrak{g}})$ yields

$$I_\mathcal{O}(\mu) = \int\limits_{\mathfrak{g}} \int\limits_{\mathscr{U}_\mathcal{O}} e^{i\psi_\mathcal{O}(u,X)/\mu}\, (\beta(\pi(u)))^{-1} a\big(\varphi_\mathcal{O}^{-1}(\pi(u)), X\big)\chi_{U_\mathcal{O}}(\varphi_\mathcal{O}^{-1}(\pi(u)))\, du\, dX \tag{5.7}$$

for any volume density du on $\mathscr{U}_{\mathcal{O}}$. In view of (5.1), we choose on $\mathscr{U}_{\mathcal{O}}$ the product measure $du = dg\,d\xi\,dv\,dw$, where dg is the Haar measure on G that we chose earlier, $d\xi$ is the Lebesgue measure on $\mathfrak{m}^*_{\mathcal{O}}$ associated to the $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$ that we fixed in Section 4.2, and dw, dv are the canonical volume forms defined by the symplectic forms in $W_{\mathcal{O}}$ and $V_{\mathcal{O},G_{\mathcal{O}}}$, respectively. By Theorem 4.10 and the Fubini theorem we can then re-write (5.7) as

$$I_{\mathcal{O}}(\mu) = \int\limits_{\mathfrak{g}}\int\limits_{G}\int\limits_{\mathscr{M}_{\mathcal{O}}}\int\limits_{\mathscr{V}_{\mathcal{O}}}\int\limits_{\mathscr{B}_{\mathcal{O}}(1/2)} e^{i\widetilde{\psi}_{\mathcal{O}}(g,\xi,w,X)/\mu}\,\widetilde{a}_{\mathcal{O}}\big(g,\xi,v,w,X\big)\,dw\,dv\,d\xi\,dg\,dX, \tag{5.8}$$

where we introduced the right-$G_{\mathcal{O}}$-invariant function $\widetilde{a}_{\mathcal{O}} \in \mathrm{C}^\infty_{\mathrm{c}}(G \times \mathscr{M}_{\mathcal{O}} \times \mathscr{V}_{\mathcal{O}} \times \mathscr{B}_{\mathcal{O}}(1/2) \times \mathfrak{g})$ given by

$$\widetilde{a}_{\mathcal{O}}\big(g,\xi,v,w,X\big) := (\beta(\pi(g,\xi,v,w)))^{-1} a\big(\varphi_{\mathcal{O}}^{-1}(\pi(g,\xi,v,w)),X\big)\chi_{U_{\mathcal{O}}}(\varphi_{\mathcal{O}}^{-1}(\pi(g,\xi,v,w))). \tag{5.9}$$

Having chosen a concrete measure on $\mathscr{U}_{\mathcal{O}}$, we now want to compute the fiber volumes $\beta(\pi(g,\xi,v,w))$ induced by this measure more explicitly in the special case that $\pi(g,\xi,v,w) \in \mathcal{J}_{\mathcal{O}}^{-1}(0)$, which is equivalent to $\xi = 0$ and $w \in \Phi_{W_{\mathcal{O}}}^{-1}(0)$. Any form η on the space $G \times \mathfrak{m}^*_{\mathcal{O}} \times V_{\mathcal{O}} = G \times \mathfrak{m}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}}$ fulfilling (5.6) is now characterized by the relation

$$\big|\mathrm{pr}^*_G(dg) \wedge \mathrm{pr}^*_{\mathfrak{m}^*_{\mathcal{O}}}(d\xi) \wedge \mathrm{pr}^*_{V_{\mathcal{O},G_{\mathcal{O}}}}(dv) \wedge \mathrm{pr}^*_{W_{\mathcal{O}}}(dw)\big| = \big|\pi^*(dy) \wedge \eta\big|, \tag{5.10}$$

where $dy = \frac{1}{(\dim \mathcal{Y}_{\mathcal{O}}/2)!}\omega_{\mathcal{Y}_{\mathcal{O}}}^{\wedge \dim \mathcal{Y}_{\mathcal{O}}/2}$ is the standard volume form on $\mathcal{Y}_{\mathcal{O}}$, the map

$$\pi : G \times \mathfrak{m}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}} \to (G \times \mathfrak{m}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}})/G_{\mathcal{O}} = \mathcal{Y}_{\mathcal{O}}$$

is the canonical projection, and $\mathrm{pr}_\bullet$ denotes the projection onto the factor $\bullet$. We only want to compute the values

$$\beta([g,0,v,w]) = \int_{\pi^{-1}([g,0,v,w])} \eta_{[g,0,v,w]} = \int_{(g,0,v,w)\cdot G_{\mathcal{O}}} |i^*_{(g,0,v,w)\cdot G_{\mathcal{O}}}\eta|,$$
$$(g,0,v,w) \in G \times \{0\} \times V_{\mathcal{O},G_{\mathcal{O}}} \times \Phi_{W_{\mathcal{O}}}^{-1}(0),$$

and for this it suffices to deduce a concrete description of the pullbacks $i^*_{(g,0,v,w)\cdot G_{\mathcal{O}}}\eta$ of the form η to the orbits $(g,0,v,w)\cdot G_{\mathcal{O}}$. In order to work out this description, we recall from (4.22) that there is a $G_{\mathcal{O}}$-equivariant diffeomorphism

$$\begin{aligned}\alpha_{\mathcal{O}} : G \times \mathfrak{m}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}} &\longrightarrow \Phi^{-1}(0) \subset G \times \mathfrak{m}^*_{\mathcal{O}} \times \mathfrak{g}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}},\\ (g,\xi,v,w) &\longmapsto (g,\xi,\Phi_{W_{\mathcal{O}}}(w),v,w),\end{aligned}$$

where we used (4.37), and that $\alpha_{\mathcal{O}}$ induces a diffeomorphism

$$\widetilde{\alpha}_{\mathcal{O}} : \mathcal{Y}_{\mathcal{O}} \xrightarrow{\cong} \Phi^{-1}(0)/G_{\mathcal{O}}$$

which was used to define the symplectic structure on $\mathcal{Y}_{\mathcal{O}}$. Thus, the symplectic form $\omega_{\mathcal{Y}_{\mathcal{O}}}$ on $\mathcal{Y}_{\mathcal{O}}$ is characterized by the property that

$$i^*\omega = \Pi^*(\widetilde{\alpha}_{\mathcal{O}}^{-1})^*\omega_{\mathcal{Y}_{\mathcal{O}}}, \tag{5.11}$$

where

$$i : \Phi^{-1}(0) \hookrightarrow G \times \mathfrak{m}^*_{\mathcal{O}} \times \mathfrak{g}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}} \cong T^*G \times V$$

denotes the inclusion,

$$\Pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G_{\mathcal{O}}$$

the canonical projection, and

$$\omega = \mathrm{pr}^*_{T^*G}\,\omega_{T^*G} + \mathrm{pr}^*_{V_{\mathcal{O},G_{\mathcal{O}}}}\,\omega_{V_{\mathcal{O},G_{\mathcal{O}}}} + \mathrm{pr}^*_{W_{\mathcal{O}}}\,\omega_{W_{\mathcal{O}}} \tag{5.12}$$

the product symplectic form on $T^*G \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}} = T^*G \times V_{\mathcal{O}}$. For a closed subgroup $H \subset G_{\mathcal{O}}$, let

$$\iota_H : G \times \{0\} \times V_{\mathcal{O},G_{\mathcal{O}}} \times \Phi_{W_{\mathcal{O}}}^{-1}(0)_{(H)} \hookrightarrow G \times \mathfrak{m}^*_{\mathcal{O}} \times V_{\mathcal{O},G_{\mathcal{O}}} \times W_{\mathcal{O}}$$

be the inclusion. Composition with $\iota_H^*\alpha_{\mathcal{O}}^*$ on both sides of (5.11) yields the relation

$$\iota_H^*\alpha_{\mathcal{O}}^* i^*\omega = \iota_H^*\alpha_{\mathcal{O}}^*\Pi^*(\widetilde{\alpha}_{\mathcal{O}}^{-1})^*\omega_{\mathcal{Y}_{\mathcal{O}}} = \iota_H^*\pi^*\omega_{\mathcal{Y}_{\mathcal{O}}}, \tag{5.13}$$

where we used that $\widetilde{\alpha}_{\mathcal{O}}^{-1}\circ\Pi\circ\alpha_{\mathcal{O}} = \pi$ holds by construction of $\widetilde{\alpha}_{\mathcal{O}}$. To compute $\iota_H^*\alpha_{\mathcal{O}}^* i^*\omega$, note that $i\circ\alpha_{\mathcal{O}}$ acts as the identity on the factors $V_{\mathcal{O},G_{\mathcal{O}}}$ and $W_{\mathcal{O}}$, so that

$$\alpha_{\mathcal{O}}^* i^*\big(\mathrm{pr}_{V_{\mathcal{O},G_{\mathcal{O}}}}^*\,\omega_{V_{\mathcal{O},G_{\mathcal{O}}}}\big) = \mathrm{pr}_{V_{\mathcal{O},G_{\mathcal{O}}}}^*\,\omega_{V_{\mathcal{O},G_{\mathcal{O}}}}, \tag{5.14}$$

$$\alpha_{\mathcal{O}}^* i^*\big(\mathrm{pr}_{W_{\mathcal{O}}}^*\,\omega_{W_{\mathcal{O}}}\big) = \mathrm{pr}_{W_{\mathcal{O}}}^*\,\omega_{W_{\mathcal{O}}}, \tag{5.15}$$

where we commited a slight abuse of notation by denoting the projections onto $V_{\mathcal{O},G_{\mathcal{O}}}$ and $W_{\mathcal{O}}$ in the spaces $G\times\mathfrak{m}_{\mathcal{O}}^*\times V_{\mathcal{O},G_{\mathcal{O}}}\times W_{\mathcal{O}}$ and $G\times\mathfrak{m}_{\mathcal{O}}^*\times\mathfrak{g}_{\mathcal{O}}^*\times V_{\mathcal{O},G_{\mathcal{O}}}\times W_{\mathcal{O}}$ by the same symbols, respectively. On the other hand, using just the definitions of $\alpha_{\mathcal{O}}$, ι_H, and the standard symplectic form ω_{T^*G}, it is straightforward to show that

$$\iota_H^*\alpha_{\mathcal{O}}^* i^*\big(\mathrm{pr}_{T^*G}^*\,\omega_{T^*G}\big) = 0. \tag{5.16}$$

Let now $(g,0,v,w)\in G\times\{0\}\times V_{\mathcal{O},G_{\mathcal{O}}}\times\Phi_{W_{\mathcal{O}}}^{-1}(0)_{(H_w)}$ be a point, where H_w is the isotropy group of w. Using the chosen $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$ and some new inner products in $V_{\mathcal{O},G_{\mathcal{O}}}$ and $W_{\mathcal{O}}$ whose euclidean measures agree with those defined by the symplectic volume forms, fix a frame $\{\nu_1,\dots,\nu_N\}$ of the normal bundle of the orbit $(g,0,v,w)\cdot G_{\mathcal{O}}$, consisting of the orthonormal complements of the tangent spaces of the orbits $T_{(g',0,v',w')}((g,0,v,w)\cdot G_{\mathcal{O}})$ in the ambient tangent spaces

$$T_{(g',0,v',w')}(G\times\mathfrak{m}_{\mathcal{O}}^*\times V_{\mathcal{O},G_{\mathcal{O}}}\times W_{\mathcal{O}})\cong\mathfrak{g}\times\mathfrak{m}_{\mathcal{O}}^*\times V_{\mathcal{O},G_{\mathcal{O}}}\times W_{\mathcal{O}}.$$

Let $\{A_1,\dots,A_{\dim\mathfrak{g}_{\mathcal{O}}}\}$ be an orthonormal basis of $\mathfrak{g}_{\mathcal{O}}$. Then

$$dg\llcorner(\nu_{i_1},\dots,\nu_{i_{\dim\mathfrak{m}_{\mathcal{O}}}}) = dA_1\wedge\cdots dA_{\dim\mathfrak{g}_{\mathcal{O}}}, \tag{5.17}$$

where the symbol $\llcorner$ denotes contraction and $\{\nu_{i_1},\dots,\nu_{i_{\dim\mathfrak{m}_{\mathcal{O}}}}\}$ is an appropriate subset of $\{\nu_1,\dots,\nu_N\}$. Using the compatibility of wedge products with pullbacks, and the dimension formula

$$\dim\mathcal{Y}_{\mathcal{O}} = \dim V_{\mathcal{O},G_{\mathcal{O}}}+\dim W_{\mathcal{O}}+\dim\mathfrak{m}_{\mathcal{O}}^*+\dim G-\dim G_{\mathcal{O}} = \dim V_{\mathcal{O},G_{\mathcal{O}}}+\dim W_{\mathcal{O}}+2\dim\mathfrak{m}_{\mathcal{O}}^*$$

which holds by construction of the model space $\mathcal{Y}_{\mathcal{O}}$, one can deduce from (5.10-5.17) by contracting with the frame $\{\nu_1,\dots,\nu_N\}$ the relation

$$\big|i_{(g,0,v,w)\cdot G_{\mathcal{O}}}^*\eta\big| = \big|\mathrm{pr}_G^*(dA_1\wedge\cdots\wedge dA_{\dim\mathfrak{g}_{\mathcal{O}}})\big|, \tag{5.18}$$

where we denoted the projection from the orbit $(g,0,v,w)\cdot G_{\mathcal{O}}\subset G\times\{0\}\times V_{\mathcal{O}}\times W_{\mathcal{O}}$ onto G again simply by pr_G. This means that the pullback of η to the orbit $(g,0,v,w)\cdot G_{\mathcal{O}}$ can be identified with the form $dG_{\mathcal{O}} = dA_1\wedge\cdots\wedge dA_{\dim g_{\mathcal{O}}}$ that is compatible with the Haar measure dg on G by (5.17). Identifying $g\cdot G_{\mathcal{O}}$ with $G_{\mathcal{O}}$, we conclude that one has the simple equality

$$\beta([g,0,v,w]) = \int_{\pi^{-1}\{[g,0,v,w]\}}\big|i_{(g,0,v,w)\cdot G_{\mathcal{O}}}^*\eta\big| = \int_{G_{\mathcal{O}}} dG_{\mathcal{O}} = \mathrm{vol}\,G_{\mathcal{O}}\quad\forall\,(g,v,w): w\in\Phi_{W_{\mathcal{O}}}^{-1}(0). \tag{5.19}$$

Let us now return to the computation of the integral $I_{\mathcal{O}}(\mu)$. Recall that dX is the Lebesgue measure on $\mathfrak{g}$ associated to the $\mathrm{Ad}\,(G)$-invariant inner product that we fixed in Section 4.2. Using the splitting

(4.40) of the reduced model phase function and substituting $X = \mathrm{Ad}\,(g)^{-1}(X')$, we can write

$$
\begin{aligned}
I_{\mathcal{O}}(\mu) &= \int\limits_{G}\int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{m}^*_{\mathcal{O}}}\int\limits_{W_{\mathcal{O}}}\int\limits_{V_{\mathcal{O},G_{\mathcal{O}}}} e^{i\left[\psi_{\mathfrak{m}^*_{\mathcal{O}}}(\xi,\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}}}\circ\mathrm{Ad}\,(g)X)+\psi_{W_{\mathcal{O}}}(w,\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}}}\circ\mathrm{Ad}\,(g)X)\right]/\mu}\, \widetilde{a}_{\mathcal{O}}(g,\xi,w,v,X)\, dv\, dw\, d\xi\, dX\, dg\\
&= \int\limits_{G}\int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{m}^*_{\mathcal{O}}}\int\limits_{W_{\mathcal{O}}}\int\limits_{V_{\mathcal{O},G_{\mathcal{O}}}} e^{i\left[\psi_{\mathfrak{m}^*_{\mathcal{O}}}(\xi,\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}}}X)+\psi_{W_{\mathcal{O}}}(w,\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}}}X)\right]/\mu}\\
&\qquad\qquad \cdot\, \widetilde{a}_{\mathcal{O}}\big(g,\xi,w,v,\mathrm{Ad}\,(g)^{-1}(X)\big)\, \underbrace{\left|\det \mathrm{Ad}\,(g)^{-1}\right|}_{=1}\, dv\, dw\, d\xi\, dX\, dg\\
&= \int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{m}^*_{\mathcal{O}}}\int\limits_{W_{\mathcal{O}}} e^{i\left[\psi_{\mathfrak{m}^*_{\mathcal{O}}}(\xi,\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}}}X)+\psi_{W_{\mathcal{O}}}(w,\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}}}X)\right]/\mu}\\
&\qquad\qquad \cdot \underbrace{\int\limits_{G}\int\limits_{V_{\mathcal{O},G_{\mathcal{O}}}} \widetilde{a}_{\mathcal{O}}\big(g,\xi,w,v,\mathrm{Ad}\,(g)^{-1}(X)\big)\, dv\, dg}_{=:a_{\mathcal{O}}(\xi,w,X)}\, dw\, d\xi\, dX\\
&= \int\limits_{\mathfrak{g}}\int\limits_{\mathfrak{m}^*_{\mathcal{O}}}\int\limits_{W_{\mathcal{O}}} e^{i\left[\psi_{\mathfrak{m}^*_{\mathcal{O}}}(\xi,\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}}}X)+\psi_{W_{\mathcal{O}}}(w,\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}}}X)\right]/\mu}\, a_{\mathcal{O}}(\xi,w,X)\, dw\, d\xi\, dX.
\end{aligned}
$$

To simplify the integral further, write $\mathfrak{g} = \mathfrak{m}_{\mathcal{O}} \times \mathfrak{g}_{\mathcal{O}}$, $X = A + B$ with $A \in \mathfrak{g}_{\mathcal{O}}$, $B \in \mathfrak{m}_{\mathcal{O}}$. Then, inserting the definition of $\psi_{\mathfrak{m}^*_{\mathcal{O}}}$, the oscillatory integral becomes

$$
I_{\mathcal{O}}(\mu) = \int\limits_{\mathfrak{g}_{\mathcal{O}}}\int\limits_{W_{\mathcal{O}}} e^{i\psi_{W_{\mathcal{O}}}(w,A)/\mu} \int\limits_{\mathfrak{m}_{\mathcal{O}}}\int\limits_{\mathfrak{m}^*_{\mathcal{O}}} e^{i\langle\xi,B\rangle/\mu}\, a_{\mathcal{O}}(\xi,w,A+B)\, d\xi\, dB\ \, dw\, dA, \tag{5.20}
$$

$$
a_{\mathcal{O}} \in \mathrm{C}^\infty_{\mathrm{c}}(\mathfrak{m}^*_{\mathcal{O}} \times W_{\mathcal{O}} \times \mathfrak{g}), \qquad \mathrm{supp}\; a_{\mathcal{O}} \subset \mathrm{pr}_{\mathfrak{m}^*_{\mathcal{O}}\times W_{\mathcal{O}}\times\mathfrak{g}}(\mathrm{supp}\; \widetilde{a}_{\mathcal{O}}).
$$

Considering the inner integral in (5.20), note that $(\xi, B) \mapsto \langle\xi, B\rangle$ is a very standard phase function with exactly one critical point at the origin, and this critical point is non-degenerate. More precisely, by [10, Lemma 7.7.3], one has for each $K \in \mathbb{N}$ the following estimate in the limit $\mu \to 0^+$:

$$
\begin{aligned}
\int\limits_{\mathfrak{m}_{\mathcal{O}}}\int\limits_{\mathfrak{m}^*_{\mathcal{O}}} e^{i\langle\xi,B\rangle/\mu}\, a_{\mathcal{O}}(\xi,w,A+B)\, d\xi\, dB &= (2\pi\mu)^{\dim\,\mathfrak{m}_{\mathcal{O}}}\Bigg[\sum_{k=0}^{K-1}\frac{(i\mu)^k}{k!}\,\langle\partial_B,\partial_\xi\rangle^k\, a_{\mathcal{O}}(0,w,A)\\
&\quad +\ \mathrm{O}\Bigg(\mu^K \sum_{|\alpha|\le 2K+\dim\,\mathfrak{m}_{\mathcal{O}}+1}\left\|\partial^\alpha_{B,\xi} a_{\mathcal{O}}(\cdot,w,A+\cdot)\right\|_{\mathrm{L}^2(\mathfrak{m}_{\mathcal{O}}\times\mathfrak{m}^*_{\mathcal{O}})}\Bigg)\Bigg],
\end{aligned} \tag{5.21}
$$

where the remainder estimate is uniform in $a_{\mathcal{O}}$, and $\partial^\alpha_{B,\xi} = \frac{\partial^{\alpha_1}}{\partial B_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}}}}}{\partial B_{\dim\,\mathfrak{m}_{\mathcal{O}}}^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}}}}}\frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}}}}}{\partial \xi_{\dim\,\mathfrak{m}_{\mathcal{O}}}^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}}}}}$. Consequently, setting

$$
a^k_{\mathcal{O}}(w,A) := \langle\partial_B,\partial_\xi\rangle^k\, a_{\mathcal{O}}(0,w,A), \tag{5.22}
$$

we have for each $K \in \mathbb{N}$ the estimate as $\mu \to 0^+$

$$
\begin{aligned}
(2\pi\mu)^{-\dim\,\mathfrak{m}_{\mathcal{O}}}\, I_{\mathcal{O}}(\mu) &= \sum_{k=0}^{K-1}\frac{(i\mu)^k}{k!}\int\limits_{\mathfrak{g}_{\mathcal{O}}}\int\limits_{W_{\mathcal{O}}} e^{i\psi_{W_{\mathcal{O}}}(w,A)/\mu}\, a^k_{\mathcal{O}}(w,A)\, dw\, dA\\
&+ \mathrm{O}_{\mathcal{O}}\Bigg(\mu^K \mathrm{vol}\big(\mathrm{pr}_{W_{\mathcal{O}}\times\mathfrak{g}_{\mathcal{O}}}(\mathrm{supp}\; a_{\mathcal{O}})\big) \sup_{(w,A)\in W_{\mathcal{O}}\times\mathfrak{g}_{\mathcal{O}}}\ \sum_{|\alpha|\le 2K+\dim\,\mathfrak{m}_{\mathcal{O}}+1}\left\|\partial^\alpha_{B,\xi} a_{\mathcal{O}}(\cdot,w,A+\cdot)\right\|_{\mathrm{L}^2(\mathfrak{m}_{\mathcal{O}}\times\mathfrak{m}^*_{\mathcal{O}})}\Bigg),
\end{aligned} \tag{5.23}
$$

where the implicit constants in the estimate are independent of K and $a_\mathcal{O}$, hence of a, but they depend of course on $\mathcal{O}$, as indicated. Hence, everything has been reduced to the question of describing the asymptotic behavior of the oscillatory integral

$$I^k_\mathcal{O}(\mu) := \int_{\mathfrak{g}_\mathcal{O}} \int_{W_\mathcal{O}} e^{i\psi_{W_\mathcal{O}}(w,A)/\mu}\, a^k_\mathcal{O}(w,A)\, dw\, dA, \qquad \mu > 0, \quad k = 0,1,2,\dots, \tag{5.24}$$

with the standard phase function

$$\psi_{W_\mathcal{O}}(w,A) = \Phi_{W_\mathcal{O}}(w)(A)$$

and an amplitude that fulfills

$$a^k_\mathcal{O} \in \mathrm{C}^\infty_\mathrm{c}(W_\mathcal{O} \times \mathfrak{g}_\mathcal{O}), \qquad \operatorname{supp} a^k_\mathcal{O} \subset \mathrm{pr}_{W_\mathcal{O}\times\mathfrak{g}_\mathcal{O}}(\operatorname{supp} a_\mathcal{O}). \tag{5.25}$$

Remark 5.1. Note that by a theorem of Weinstein, a symplectic manifold is locally the cotangent bundle of a Lagrangian submanifold. But in general, the momentum map of a Hamiltonian action will not be given locally by a Liouville form, and the situation here constitutes an example of such a Hamiltonian action on a cotangent bundle that is not simply a lift of a group action on the basis manifold. In particular, the results in [21] do not imply asymptotics for the integrals in (5.24).

Now, it is useful to distinguish between two cases:

$\boxed{\textbf{Case 1: } \psi_{W_\mathcal{O}} \equiv 0.}$ This occurs precisely if $W_\mathcal{O} = \{0\}$ or $\mathfrak{g}_\mathcal{O} = \{0\}$, so that all orbits in $W_\mathcal{O}$ have the same dimension (either $\dim G_\mathcal{O}$ or 0). The integral (5.24) then evaluates trivially to

$$I^k_\mathcal{O}(\mu) = \int_{\mathfrak{g}_\mathcal{O}} \int_{W_\mathcal{O}} a^k_\mathcal{O}(w,A)\, dw\, dA =: \mathcal{L}_{\mathcal{O},k}, \qquad \mu > 0, \quad k = 0,1,2,\dots, \tag{5.26}$$

in particular it is independent of μ and contributes to (5.23) only as a factor in the coefficients. For later use, let us compute the 0-th order coefficient $\mathcal{L}_{\mathcal{O},0}$ explicitly. By (5.9), (5.22), and (5.19), it is given by

$$\begin{aligned}
\mathcal{L}_{\mathcal{O},0} &= \int_{\mathfrak{g}_\mathcal{O}} \int_{W_\mathcal{O}} a^0_\mathcal{O}(w,A)\, dw\, dA = \int_{\mathfrak{g}_\mathcal{O}} \int_{W_\mathcal{O}} a_\mathcal{O}(0,w,A)\, dw\, dA \\
&= \int_{\mathfrak{g}_\mathcal{O}} \int_G \int_{W_\mathcal{O}} \int_{V_{\mathcal{O},G_\mathcal{O}}} \widetilde{a}_\mathcal{O}\big(g,0,w,v,\mathrm{Ad}\,(g)^{-1}(A)\big)\, dv\, dw\, dg\, dA \\
&= (\operatorname{vol} G_\mathcal{O})^{-1} \int_{W_\mathcal{O}} \int_{V_{\mathcal{O},G_\mathcal{O}}} \int_G \int_{\mathfrak{g}_\mathcal{O}} a\big(\varphi_\mathcal{O}^{-1}(\pi(g,0,v,0)), \mathrm{Ad}\,(g)^{-1}(A)\big)\, dA \chi_{U_\mathcal{O}}(\varphi_\mathcal{O}^{-1}(\pi(g,0,v,w)))\, dg\, dv\, dw.
\end{aligned}$$

Observe that, since $\psi_{W_\mathcal{O}} \equiv 0$, we have $\Phi^{-1}_{W_\mathcal{O}}(0) = W_\mathcal{O}$, $\Phi^{-1}_{V_\mathcal{O}}(0) = V_\mathcal{O}$. Using (4.29), (4.38), and (4.46), we can therefore transform the integral into

$$\begin{aligned}
\mathcal{L}_{\mathcal{O},0} = (\operatorname{vol} G_\mathcal{O})^{-1} \sum_{j=1}^{N_\mathcal{O}} \int_{(\Omega_{M(G^j_\mathcal{O})} \cap U_\mathcal{O}/G)} \int_G \int_{\mathfrak{g}_\mathcal{O}} a\big(\varphi_\mathcal{O}^{-1}(\pi(g,0,\widetilde{\varphi}_\mathcal{O}(\mathcal{O}'),0)), \mathrm{Ad}\,(g)^{-1}(A)\big)\, dA \\
\chi_{U_\mathcal{O}}(\varphi_\mathcal{O}^{-1}(\pi(g,0,\widetilde{\varphi}_\mathcal{O}(\mathcal{O}'),0)))\, dg\, d(\Omega_{M(G^j_\mathcal{O})}/G)(\mathcal{O}'),
\end{aligned}$$

where

$$(G_\mathcal{O}) = (G^1_\mathcal{O}), \dots, (G^{N_\mathcal{O}}_\mathcal{O}), \qquad N_\mathcal{O} \in \mathbb{N},$$

are the isotropy types occurring in $W_\mathcal{O}$. If $W_\mathcal{O} = \{0\}$, the list above consists only of the single element $(G_\mathcal{O})$, and if $W_\mathcal{O} \neq \{0\}$, we have $g_\mathcal{O} = \{0\}$, so that the list can contain several isotropy types but then all of them are represented by *finite* groups. For $j \in \{1,\dots,N_\mathcal{O}\}$, let $\mathcal{L}^j_{\mathcal{O},0}$ denote the j-th summand in $\mathcal{L}_{\mathcal{O},0}$. With the notation

$$p_\mathcal{O}(\mathcal{O}') := \varphi_\mathcal{O}^{-1}(\pi(e,0,\widetilde{\varphi}_\mathcal{O}(\mathcal{O}'),0)) \in \mathcal{O}' \subset U_\mathcal{O}$$

and the observation

$$g\cdot p_{\mathcal{O}}(\mathcal{O}')=g\cdot\varphi_{\mathcal{O}}^{-1}(\pi(e,0,\widetilde{\varphi}_{\mathcal{O}}(\mathcal{O}'),0))=\varphi_{\mathcal{O}}^{-1}(\pi(g,0,\widetilde{\varphi}_{\mathcal{O}}(\mathcal{O}'),0)),$$

we obtain for $\mathcal{L}^j_{\mathcal{O},0}$ the expression

$$(\operatorname{vol}G_{\mathcal{O}})^{-1}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathfrak{g}_{\mathcal{O}}} a\big(g\cdot p_{\mathcal{O}}(\mathcal{O}'),\mathrm{Ad}\,(g)^{-1}(A)\big)\,dA\chi_{U_{\mathcal{O}}}(g\cdot p_{\mathcal{O}}(\mathcal{O}'))\,dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}').$$

To proceed, recall from the construction of the space $\mathcal{Y}_{\mathcal{O}}$ that the Lie algebra $\mathfrak{g}_{\mathcal{O}}$ is precisely the stabilizer algebra of some chosen point $p_{\mathcal{O}}(\mathcal{O})\in\mathcal{O}$. Now, for any point $p\in\mathcal{O}$ and any Lie group element $g\in G$, we have

$$h\in G_{\mathcal{O}\,g\cdot p}\iff h\cdot g\cdot p=g\cdot p\iff (g^{-1}hg)\cdot p=p,$$

which shows that $G_{g\cdot p}=g^{-1}G_{\mathcal{O}p}g$ and hence $\mathfrak{g}_{g\cdot p}=\mathrm{Ad}\,(g^{-1})(\mathfrak{g}_p)=\mathrm{Ad}\,(g)^{-1}(\mathfrak{g}_p)$. Moreover, we know from (4.45) that $G_{p_{\mathcal{O}}(\mathcal{O}')}=G_{p_{\mathcal{O}}(\mathcal{O})}=G^j_{\mathcal{O}}$ for all $\mathcal{O}'\in(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}})/G$. Using these observations, we arrive at

$$\begin{aligned}\mathcal{L}^j_{\mathcal{O},0}&=(\operatorname{vol}G_{\mathcal{O}})^{-1}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathrm{Ad}\,(g)^{-1}(\mathfrak{g}_{\mathcal{O}})} a\big(g\cdot p_{\mathcal{O}}(\mathcal{O}'),A\big)\,dA\,\underbrace{|\det\,\mathrm{Ad}\,(g)|_{\mathrm{Ad}\,(g)^{-1}(h_{\mathcal{O}})}|}_{=1}\\&\qquad\chi_{U_{\mathcal{O}}}(g\cdot p_{\mathcal{O}}(\mathcal{O}'))\,dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}')\\&=(\operatorname{vol}G_{\mathcal{O}})^{-1}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathfrak{g}_{g\cdot p_{\mathcal{O}}(\mathcal{O}')}} a\big(g\cdot p_{\mathcal{O}}(\mathcal{O}'),A\big)\,dA\,\chi_{U_{\mathcal{O}}}(g\cdot p_{\mathcal{O}}(\mathcal{O}'))\,dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}').\end{aligned}$$

Finally, suppose that we replace $p_{\mathcal{O}}(\mathcal{O}')$ by a different point $p'(\mathcal{O}')$ in the orbit $\mathcal{O}'$. Then $g'(\mathcal{O}')\cdot p_{\mathcal{O}}(\mathcal{O}')=p'(\mathcal{O}')$ for some $g'(\mathcal{O}')\in G$. The integral obtained by using $p'(\mathcal{O}')$ instead of $p_{\mathcal{O}}(\mathcal{O}')$ then reads

$$\begin{aligned}\mathcal{L}^{j\,\prime}_{\mathcal{O},0}&=(\operatorname{vol}G_{\mathcal{O}})^{-1}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathfrak{g}_{g\cdot p'(\mathcal{O}')}} a\big(g\cdot p'(\mathcal{O}'),A\big)\,dA\,\chi_{U_{\mathcal{O}}}(g\cdot p'(\mathcal{O}'))\,dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}')\\&=(\operatorname{vol}G_{\mathcal{O}})^{-1}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathfrak{g}_{g\cdot g'(\mathcal{O}')\cdot p_{\mathcal{O}}(\mathcal{O}')}} a\big(g\cdot g'(\mathcal{O}')\cdot p_{\mathcal{O}}(\mathcal{O}'),A\big)\,dA\,\chi_{U_{\mathcal{O}}}(g\cdot g'(\mathcal{O}')\cdot p_{\mathcal{O}}(\mathcal{O}'))\\&\qquad dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}')\\&=(\operatorname{vol}G_{\mathcal{O}})^{-1}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathfrak{g}_{g\cdot p_{\mathcal{O}}(\mathcal{O}')}} a\big(g\cdot p_{\mathcal{O}}(\mathcal{O}'),A\big)\,dA\,\chi_{U_{\mathcal{O}}}(g\cdot p_{\mathcal{O}}(\mathcal{O}'))\,dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}')\\&=\mathcal{L}^j_{\mathcal{O},0},\end{aligned}$$

where we used the right-G-invariance of the Haar measure on G. Thus, in the case $\psi_{W_{\mathcal{O}}}\equiv 0$, we conclude that the leading term in the integral (5.23) is explicitly given by

$$\mathcal{L}_{\mathcal{O},0}=(\operatorname{vol}G_{\mathcal{O}})^{-1}\sum_{j=1}^{N_{\mathcal{O}}}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}\cap U_{\mathcal{O}}/G)}\int\limits_G\int\limits_{\mathfrak{g}_{g\cdot p(\mathcal{O}')}} a\big(g\cdot p(\mathcal{O}'),A\big)\,dA\,\chi_{U_{\mathcal{O}}}(g\cdot p(\mathcal{O}'))\,dg\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}'),$$

where $p(\mathcal{O}')\in\mathcal{O}'$ is an arbitrary base point in the orbit $\mathcal{O}'$. Using the notation (4.16), and recalling that $\chi_{U_{\mathcal{O}}}$ is compactly supported in $U_{\mathcal{O}}$, we can write the result more compactly as

$$\mathcal{L}_{\mathcal{O},0}=\frac{\operatorname{vol}G}{\operatorname{vol}G_{\mathcal{O}}}\sum_{j=1}^{N_{\mathcal{O}}}\int\limits_{(\Omega_{M(G^j_{\mathcal{O}})}/G)}\;\fint\limits_{\mathcal{O}'}\int\limits_{\mathfrak{g}_p} a(p,A)\,dA\,\chi_{U_{\mathcal{O}}}(p)\,d\mathcal{O}'(p)\,d(\Omega_{M(G^j_{\mathcal{O}})}/G)(\mathcal{O}'). \tag{5.27}$$

This expression is independent from our choices of measures on G and $G_{\mathcal{O}}$ (as long as they are $G_{\mathcal{O}}$-invariant), and the only reminescence of the desingularization process is the presence of the cutoff function $\chi_{U_{\mathcal{O}}}$. Since in the present case $W_{\mathcal{O}} \neq \{0\}$ implies $\mathfrak{g}_{\mathcal{O}} = \{0\}$, in which case only orbifold singularities occur in $U_{\mathcal{O}}$, we can write (5.27) also as

(5.28)
$$\mathcal{L}_{\mathcal{O},0} = \frac{\operatorname{vol} G}{\operatorname{vol} G_{\mathcal{O}}} \cdot \begin{cases} \displaystyle\int_{(\Omega_{M(G_{\mathcal{O}})}/G)} \fint_{\mathcal{O}'} \int_{\mathfrak{g}_p} a(p,A)\, dA\, \chi_{U_{\mathcal{O}}}(p)\, d\mathcal{O}'(p)\, d(\Omega_{M(G_{\mathcal{O}})}/G)(\mathcal{O}'), & W_{\mathcal{O}} = \{0\}, \\ \displaystyle\int_{(U_{\mathcal{O}}/G)} \fint_{\mathcal{O}'} a(p,0) \chi_{U_{\mathcal{O}}}(p)\, d\mathcal{O}'(p)\, d(U_{\mathcal{O}}/G)(\mathcal{O}'), & g_{\mathcal{O}} = \{0\}, \end{cases}$$

where in the second line $d(U_{\mathcal{O}}/G)$ is the orbifold measure induced on the orbit space $U_{\mathcal{O}}/G$ by $dM|_{U_{\mathcal{O}}}$.

$\boxed{\textbf{Case 2: } \psi_{W_{\mathcal{O}}} \not\equiv 0.}$ In this case, one necessarily has $W_{\mathcal{O}} \neq \{0\}$, orbits of several different types and possibly different dimensions occur in $U_{\mathcal{O}}$ (see Corollary 4.13), and one cannot directly compute the oscillatory integral (5.24) in general. We therefore proceed by resolving the singularities of the critical set of the phase function $\psi_{W_{\mathcal{O}}}$.

5.3. **Discretized blow-up in the symplectic vector space.** In the non-trivial case $\psi_{W_{\mathcal{O}}} \not\equiv 0$ we shall now proceed to a partial desingularization of the critical set of the phase function $\psi_{W_{\mathcal{O}}}(x, A)$ in order to obtain a clean monomialization of the latter in the sense of Section 3. The central idea is to successively blow up the strata of maximal singular orbits, beginning by performing the blow-up

$$\Pi_{W_{\mathcal{O}}} : B_{Z_{\mathcal{O}}} W_{\mathcal{O}} \longrightarrow W_{\mathcal{O}}$$

of $W_{\mathcal{O}}$ with center $Z_{\mathcal{O}} = \{w = 0\}$, which represents the most singular $G_{\mathcal{O}}$-orbit. To do so, set $N := \dim W_{\mathcal{O}}$ (note that $N \geq 2$), and fix a basis for $W_{\mathcal{O}}$, which is equivalent to introducing coordinates $(w_1, \dots, w_N)$, so that $Z_{\mathcal{O}}$ is given in terms of the common zero set of these coordinate functions. The blow-up of $W_{\mathcal{O}}$ with center $Z_{\mathcal{O}}$ is then given by[5]

$$B_{Z_{\mathcal{O}}} W_{\mathcal{O}} := \Big\{(w, [t]) \in W_{\mathcal{O}} \times \mathbb{RP}^{N-1} : \quad t_i w_j = w_i t_j\Big\},$$

where $[t] = [t_1, \dots, t_N]$ are homogeneous coordinates in $\mathbb{RP}^{N-1}$, and

$$\Pi_{W_{\mathcal{O}}} : B_{Z_{\mathcal{O}}} W_{\mathcal{O}} \to W_{\mathcal{O}}, \qquad (w, [t]) \longmapsto w.$$

$\Pi_{W_{\mathcal{O}}}$ is a surjective, proper mapping, and $\Pi^{-1}(Z_{\mathcal{O}}) \simeq \mathbb{RP}^{N-1}$ is called the *exceptional divisor*. To give a local description, one covers $B_{Z_{\mathcal{O}}} W_{\mathcal{O}}$ with the open sets

$$\widetilde{\mathscr{U}_\varrho} := \{(w, [t]) \in B_{Z_{\mathcal{O}}} W_{\mathcal{O}} : t_\varrho \neq 0\}, \qquad \varrho \in \{1, \dots, N\},$$

and introduces the coordinates

$$\gamma_\varrho : \widetilde{\mathscr{U}_\varrho} \hookrightarrow \mathbb{R}^N, \quad (w, [t]) \longmapsto \left(\frac{t_1}{t_\varrho}, \dots, \frac{t_{\varrho-1}}{t_\varrho}, \sigma_\varrho, \frac{t_{\varrho+1}}{t_\varrho}, \dots, \frac{t_N}{t_\varrho}\right) =: \tau.$$

Then, in each single chart $\widetilde{\mathscr{U}_\varrho}$ the blow-up Π reads

$$\Pi_{W_{\mathcal{O}}} \circ \gamma_\varrho^{-1} : \tau \longmapsto (\tau_\varrho(\tau_1, \dots, 1, \dots, \tau_N)) = (w_1, \dots, w_N),$$

τ_ϱ being the so-called *exceptional parameter*. The phase function $\psi_{W_{\mathcal{O}}}$ then factorizes by homogeneity simply according to

$$\psi_{W_{\mathcal{O}}}(\Pi_{W_{\mathcal{O}}} \circ \gamma_\varrho^{-1}(\tau), A) = \tau_\varrho^2\, \psi_{W_{\mathcal{O}}}((\tau_1, \dots, 1, \dots, \tau_N), A).$$

Equivalently, one can describe the blow-up in terms of polar coordinates by identifying $B_{Z_{\mathcal{O}}} W_{\mathcal{O}}$ with the quotient of $\mathbb{R} \times S^{N-1}$ by the relation $(\tau, \omega) \sim (-\tau, -\omega)$, S^{N-1} being the unit sphere in $W_{\mathcal{O}} \simeq \mathbb{R}^N$.

[5]Note that the condition $t_i \sigma_j = \sigma_i t_j$ precisely means that σ belongs to the line $[t]$.

The blow-up is then simply given by $\Pi_{W_\mathcal{O}} : [r,\omega] \longmapsto r\omega$, and the exceptional divisor by $\Pi^{-1}(Z_\mathcal{O}) = \{[0,\omega] : \omega \in S^{N-1}\}$. Furthermore, we have the factorization

$$\psi_{W_\mathcal{O}}(\Pi \circ \gamma_\varrho^{-1}([r,\omega]), A) = r^2\, \psi_{W_\mathcal{O}}(\omega, A).$$

Note that away from the center $Z_\mathcal{O} = \{w = 0\}$, the blow-up $\Pi_{W_\mathcal{O}}$ is globally described by polar coordinates. Therefore, for the treatment of the integral (5.24), it will simply suffice to introduce such coordinates in $W_\mathcal{O}$, the center $Z_\mathcal{O}$ being of measure zero. Now, for the ensuing phase analysis and the later iteration of the desingularization process, it will actually be much more convenient to introduce *radially discrete polar coordinates*. The underlying reason is that the unit sphere in $W_\mathcal{O}$ does not carry a symplectic structure for simple dimension reasons, while an open subset of $W_\mathcal{O}$ is naturally a symplectic submanifold. Therefore, we consider the decomposition of the unit ball

$$\mathscr{B}_\mathcal{O}(1) - \{0\} = \bigcup_{l=0}^{\infty} \mathscr{B}_\mathcal{O}\big(2^{-l}\big) - \mathscr{B}_\mathcal{O}\big(2^{-(l+1)}\big).$$

Let us thicken the open spherical shells $\mathscr{B}_\mathcal{O}\big(2^{-l}\big) - \overline{\mathscr{B}_\mathcal{O}\big(2^{-(l+1)}\big)}$ slightly so that they overlap. Setting

$$\mathscr{W}_\mathcal{O}^l := \mathscr{B}_\mathcal{O}\big(2^{-l}\big) - \overline{\mathscr{B}_\mathcal{O}\big(2^{-(l+1)} - 2^{-(l+2)}\big)}, \qquad l \in \{0,1,2,\dots\}$$

we get an open cover $\mathbf{W}_\mathcal{O} := \{\mathscr{W}_\mathcal{O}^l\}_{l\in\{0,1,\dots\}}$ of $\mathscr{B}_\mathcal{O}(1) - \{0\}$ that consists of $G_\mathcal{O}$-invariant sets. Notice that

$$2^l \mathscr{W}_\mathcal{O}^l = \mathscr{B}_\mathcal{O}(1) - \overline{\mathscr{B}_\mathcal{O}(1/4)} =: M_\mathcal{O}$$

is the same open $G_\mathcal{O}$-invariant set at distance $1/4$ to the origin in $W_\mathcal{O}$ for each $l \in \{0,1,2,\dots\}$. Therefore, we can find an $G_\mathcal{O}$-invariant cutoff function $\chi_{M_\mathcal{O}} \in \mathrm{C}_\mathrm{c}^\infty(M_\mathcal{O})$ whose outer slope differs from the inner slope by the factor $1/2$ such that the functions

$$\chi_{\mathscr{W}_\mathcal{O}^l}(w) := \chi_{M_\mathcal{O}}(2^l w), \qquad w \in \mathscr{W}_\mathcal{O}^l,$$

yield a partition of unity $\{\chi_{\mathscr{W}_\mathcal{O}^l}\}_{l\in\{0,1,\dots\}}$ on $\mathscr{B}_\mathcal{O}(1/2)$ subordinate to $\mathbf{W}_\mathcal{O}$. Note that each of the functions $\chi_{\mathscr{W}_\mathcal{O}^l}$ is necessarily compactly supported because $\mathscr{W}_\mathcal{O}^l$ is pre-compact in $W_\mathcal{O}$. As the origin is a set of measure zero in $\mathscr{B}_\mathcal{O}(1/2)$, and the projection of the support of $a_\mathcal{O}^k$ onto $W_\mathcal{O}$ is contained in $\mathscr{B}_\mathcal{O}(1/2)$ by assumption, we obtain with the Lebesgue theorem on bounded convergence

$$I_\mathcal{O}^k(\mu) = \sum_{l=0}^{\infty} I_\mathcal{O}^{k,l}(\mu), \qquad I_\mathcal{O}^{k,l}(\mu) := \int_{\mathfrak{g}_\mathcal{O}} \int_{\mathscr{W}_\mathcal{O}^l} e^{i\psi_{W_\mathcal{O}}(w,A)/\mu}\, a_\mathcal{O}^k(w,A)\chi_{\mathscr{W}_\mathcal{O}^l}(w)\, dw\, dA.$$

It now suffices to consider each integral $I_\mathcal{O}^{k,l}(\mu)$ individually. To this end, we set

$$\tau_l := 2^{-l}, \qquad l \in \{0,1,2,\dots\},$$

and perform in $I_\mathcal{O}^{k,l}(\mu)$ the change of variables $w \mapsto \tau_l w'$, which leads to

$$I_\mathcal{O}^{k,l}(\mu) = \tau_l^{\dim W_\mathcal{O}} \int_{\mathfrak{g}_\mathcal{O}} \int_{M_\mathcal{O}} e^{i\frac{\tau_l^2}{\mu}\psi_{W_\mathcal{O}}(w,A)}\, a_\mathcal{O}^k\big(\tau_l w, A\big)\chi_{M_\mathcal{O}}(w)\, dw\, dA.$$

Here we used the absolutely crucial fact that for every $r \in \mathbb{R}$ the standard phase function $\psi_{W_\mathcal{O}}$ factorizes by homogeneity according to

$$\psi_{W_\mathcal{O}}(rw, A) = \Phi_{W_\mathcal{O}}(rw)(A) = r^2\, \Phi_{W_\mathcal{O}}(w)(A) = r^2\, \psi_{W_\mathcal{O}}(w,A),$$

which is equivalent to a desingularization of its vanishing set. We now define for each $k, l \in \{0,1,2,\dots\}$ the *reduced amplitude*

$$a_\mathcal{O}^{k,l}(w,A) := a_\mathcal{O}^k\big(\tau_l w, A\big)\chi_{M_\mathcal{O}}(w), \qquad a_\mathcal{O}^{k,l} \in \mathrm{C}_\mathrm{c}^\infty(M_\mathcal{O} \times \mathfrak{g}_\mathcal{O}). \tag{5.29}$$

By construction, the reduced amplitudes fulfill

$$\operatorname{supp} a_\mathcal{O}^{k,l} \subset M_\mathcal{O} \times \operatorname{pr}_{\mathfrak{g}_\mathcal{O}}(\operatorname{supp} a) \qquad \forall\, l \in \{0,1,2,\dots\}, \tag{5.30}$$

where the latter is an l- and k-independent pre-compact subset of $W_{\mathcal{O}} \times \mathfrak{g}_{\mathcal{O}}$. Concerning the derivatives of $a_{\mathcal{O}}^{k,l}$, it is easy to see that their supremum norms do not grow as $l \to \infty$. Indeed, since the cutoff function $\chi_{M_{\mathcal{O}}}$ is independent of k, l, and a, and taking into account (5.22) and our initial technical assumption about the map $\varphi_{\mathcal{O}}$, we can find for any differential operator $D_{\mathcal{O}}$ acting on $\mathrm{C_c^\infty}(M_{\mathcal{O}} \times \mathfrak{g}_{\mathcal{O}})$ a constant $C_{\mathcal{O},D_{\mathcal{O}}} > 0$, independent of k, l, and the amplitude a, such that

$$\left\| D_{\mathcal{O}} a_{\mathcal{O}}^{k,l} \right\|_\infty \leq C_{\mathcal{O},D_{\mathcal{O}}} \left\| \widetilde{D}_{\mathcal{O}}^k a \right\|_\infty \qquad \forall\, k, l \in \{0,1,2,\dots\}, \tag{5.31}$$

where $\widetilde{D}_{\mathcal{O}}^k$ is a differential operator acting on $\mathrm{C_c^\infty}(M \times \mathfrak{g})$ with

$$\mathrm{order}(\widetilde{D}_{\mathcal{O}}^k) = \mathrm{order}(D_{\mathcal{O}}) + 2k, \tag{5.32}$$

and the assignment $D_{\mathcal{O}} \mapsto \widetilde{D}_{\mathcal{O}}^k$ is independent of the amplitude a and the index l. Moreover, the assignment is linear and acts as the identity on the Lie algebra derivatives. Writing

$$\psi_{M_{\mathcal{O}}} := \psi_{W_{\mathcal{O}}}|_{M_{\mathcal{O}} \times \mathfrak{g}}$$

and p instead of w for points in $M_{\mathcal{O}} \subset W_{\mathcal{O}}$, X instead of A for points in $\mathfrak{g}_{\mathcal{O}}$, we can now define for each $l \in \{0,1,2,\dots\}$ a new oscillatory integral with parameter $\nu > 0$:

$$I_{\mathcal{O}}^{k,l}(\nu) = \int\limits_{\mathfrak{g}_{\mathcal{O}}} \int\limits_{M_{\mathcal{O}}} e^{i\psi_{M_{\mathcal{O}}}(p,X)/\nu}\, a_{\mathcal{O}}^{k,l}(p,X)\, dp\, dX. \tag{5.33}$$

For each $k \in \{0,1,2,\dots\}$, one recovers $I_{\mathcal{O}}^k(\mu)$ as

$$I_{\mathcal{O}}^k(\mu) \;=\; \sum_{l=0}^{\infty} \tau_l^{\dim W_{\mathcal{O}}} I_{\mathcal{O}}^{k,l}\big(\mu \tau_l^{-2}\big), \tag{5.34}$$

and together with (5.23), (5.4), and (5.5), we obtain the following asymptotic formula for the original oscillatory integral $I(\mu)$:

$$\begin{aligned} I(\mu) = &\sum_{\mathcal{O} \in \aleph^+(M)} \left[(2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}}} \sum_{k=0}^{K_{\mathcal{O}}-1} \frac{(i\mu)^k}{k!} \mathcal{L}_{\mathcal{O},k} \;+\; R_{\mathcal{O}}^{K_{\mathcal{O}}}(\mu) \right] \\ &+ \sum_{\mathcal{O} \in \aleph^-(M)} \left[(2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}}} \sum_{k=0}^{K_{\mathcal{O}}-1} \frac{(i\mu)^k}{k!} \sum_{l=0}^{\infty} \tau_l^{\dim W_{\mathcal{O}}} I_{\mathcal{O}}^{k,l}\big(\mu\tau_l^{-2}\big) \;+\; R_{\mathcal{O}}^{K_{\mathcal{O}}}(\mu) \right] \;+\; I^{\square}(\mu), \end{aligned} \tag{5.35}$$

where $K_{\mathcal{O}} \in \mathbb{N}$ is arbitrary for each $\mathcal{O} \in \aleph(M)$, $(2\pi\mu)^{-\dim \mathfrak{m}_{\mathcal{O}}} R_{\mathcal{O}}^{K_{\mathcal{O}}}(\mu)$ is the remainder in (5.23), and we introduced the new notation

$$\aleph^+(M) := \{\mathcal{O} \in \aleph(M) : \psi_{W_{\mathcal{O}}} \equiv 0\}, \qquad \aleph^-(M) := \aleph(M) - \aleph^+(M).$$

The first step in the desingularization process is now complete. Each of the integrals $I_{\mathcal{O}}^{k,l}(\nu)$, $\mathcal{O} \in \aleph^-(M)$, is formally identical to the original integral $I(\mu)$ that we started with in (5.4). More precisely, this means that (5.33) is again an oscillatory integral with a compactly supported smooth amplitude over a product domain in which the first factor is a symplectic manifold that carries a Hamiltonian action of a compact Lie group, the second factor is the associated Lie algebra, and the phase function is given by the associated momentum map. Moreover, the support and the supremum norms of the derivatives of the new amplitudes $a_{\mathcal{O}}^{k,l}$ are controlled uniformly in k, l by the support and the corresponding supremum norms of the original amplitude a. The whole point of the desingularization process is that for each $\mathcal{O} \in \aleph^-(M)$, the "worst" orbit type occurring in $U_{\mathcal{O}}$ does no longer occur in $M_{\mathcal{O}}$. More precisely, one has

Proposition 5.2. *The $G_{\mathcal{O}}$-isotropy types occurring in $M_{\mathcal{O}}$ correspond to a subset of the set of those G-isotropy types in $U_{\mathcal{O}}$ which are strictly greater than $(G_{\mathcal{O}})$. The $G_{\mathcal{O}}$-isotropy types occurring in $\Omega_{M_{\mathcal{O}}}$ correspond one-to-one to the G-isotropy types in $U_{\mathcal{O}} \cap \Omega_M$ which are strictly greater than $(G_{\mathcal{O}})$, and the regular $G_{\mathcal{O}}$-isotropy type in any connected component of $\Omega_{M_{\mathcal{O}}}$ agrees with the unique regular G-isotropy type in the connected components of $U_{\mathcal{O}} \cap \Omega_M$ when considered as a G-isotropy type.*

Proof. Since $M_{\mathcal{O}}$ is an open $G_{\mathcal{O}}$-invariant subset of $W_{\mathcal{O}} - \{0\}$, the statements follow directly from Corollary 4.13 and Proposition 4.14. □

To close this section, let us emphasize three principles that are important in the desingularization process:

(1) Formal reproduction of the initial data − one can use the output again as an input for a further desingularization.
(2) Reduction of the number of occurring isotropy or orbit types − both in the manifold itself as well as in the zero level of the momentum map.
(3) Inheritance of the regular isotropy types in the zero level of the momentum map from the input to the output.

Since only a finite number of orbit types occur in $U_{\mathcal{O}}$, we need only repeat the desingularization process finitely many times until it ends automatically in *Case 1.*

6. Iteration of the desingularization process

In Section 5, we have described a general desingularization process that formally reproduces its input $I(\mu)$ in a "desingularized version": The *essential output* consists of the integrals $I^{k,l}_{\mathcal{O}}(\nu)$, where $k, l \in \{0, 1, \dots\}$, $\mathcal{O} \in \aleph^-(M)$. In particular, we can iterate the desingularization process, using the essential output of the N-th iteration as the input for the $(N + 1)$-th iteration.

6.1. **Second and third iterations.** To indicate that the desingularization process carried out in Section 5 was the first one, we now write $\mathcal{O}_1$, l_1, k_1, ν_1 instead of $\mathcal{O}$, l, k, and ν. For each $\mathcal{O}_1 \in \aleph^-(M)$ we can apply the desingularization process again to the symplectic $G_{\mathcal{O}_1}$-manifold $M_{\mathcal{O}_1}$ and each of the oscillatory integrals $I^{k_1,l_1}_{\mathcal{O}_1}(\nu_1)$ indexed by l_1 and k_1. The orbit type $(G_{\mathcal{O}_1})$ no longer occurs in $M_{\mathcal{O}_1}$, and by Corollary 4.13 we also know that the $G_{\mathcal{O}_1}$-orbit types occurring in $\Omega_{M_{\mathcal{O}_1}}$ are precisely the G-orbit types occurring in $U_{\mathcal{O}} \cap \Omega_M$ except $(G_{\mathcal{O}_1})$. By construction, the index set $\aleph(M_{\mathcal{O}_1})$ contains only orbits in $\Omega_{M_{\mathcal{O}_1}}$, so that the $G_{\mathcal{O}_1}$-orbit types associated to the orbits $\mathcal{O}_2 \in \aleph(M_{\mathcal{O}_1})$ are precisely the G-orbit types associated to the orbits $\mathcal{O}'_1 \in \aleph^-(M)$ with $\mathcal{O}'_1 \in U_{\mathcal{O}_1}$, except $(G_{\mathcal{O}_1})$. Applying the desingularization process to $I^{k_1,l_1}_{\mathcal{O}_1}(\nu_1)$ yields the formula

$$
\begin{aligned}
I^{k_1,l_1}_{\mathcal{O}_1}(\nu_1) = &\sum_{\mathcal{O}_2 \in \aleph^+(M_{\mathcal{O}_1})} \Bigg[(2\pi\nu_1)^{\dim \mathfrak{m}_{\mathcal{O}_2}} \sum_{k_2=0}^{K_{\mathcal{O}_2}} \frac{(i\nu_1)^{k_2}}{k_2!} \mathcal{L}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2,k_2} \; + \; R^{K_{\mathcal{O}_2}+1}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2}(\nu_1) \Bigg] \\
&+ \sum_{\mathcal{O}_2 \in \aleph^-(M_{\mathcal{O}_1})} \Bigg[(2\pi\nu_1)^{\dim \mathfrak{m}_{\mathcal{O}_2}} \sum_{k_2=0}^{K_{\mathcal{O}_2}} \frac{(i\nu_1)^{k_2}}{k_2!} \sum_{l_2=0}^{\infty} \tau_{l_2}^{\dim W_{\mathcal{O}_2}} I^{k_1,l_1,k_2,l_2}_{\mathcal{O}_1,\mathcal{O}_2}\big(\nu_1 \tau_{l_2}^{-2}\big) \\
&+ R^{K_{\mathcal{O}_2}+1}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2}(\nu_1) \Bigg] \; + \; I^{\square\, k_1,l_1}_{\mathcal{O}_1}(\nu_1),
\end{aligned}
\tag{6.1}
$$

where $K_{\mathcal{O}_2} \in \{0,1,2\ldots\}$ is arbitrary for each $\mathcal{O}_2 \in \aleph(M_{\mathcal{O}_1})$. Plugging (6.1) into (5.35) and collecting the terms, we arrive at the following result after the second iteration:

$$\begin{aligned}
I(\mu) = &\sum_{\mathcal{O}_1\in\aleph^+(M)} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_1}} \sum_{k_1=0}^{K_{\mathcal{O}_1}} \frac{(i\mu)^{k_1}}{k_1!}\mathcal{L}_{\mathcal{O}_1,k_1} \;+\; \sum_{\mathcal{O}_1\in\aleph(M)} R^{K_{\mathcal{O}_1}+1}_{\mathcal{O}_1}(\mu) \;+\; I^{\square}(\mu)\\
&+ \sum_{\mathcal{O}_1\in\aleph^-(M)} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_1}} \sum_{k_1=0}^{K_{\mathcal{O}_1}}\sum_{l_1=0}^{\infty} \frac{(i\mu)^{k_1}}{k_1!}\tau_{l_1}^{\dim W_{\mathcal{O}_1}} \Bigg[\sum_{\mathcal{O}_2\in\aleph^+(M_{\mathcal{O}_1})} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_2}}\\
&\cdot \sum_{k_2=0}^{K_{\mathcal{O}_2}} \frac{(i\mu)^{k_2}}{k_2!}\tau_{l_1}^{-2(\dim \mathfrak{m}_{\mathcal{O}_2}+k_2)}\mathcal{L}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2,k_2} + \sum_{\mathcal{O}_2\in\aleph(M_{\mathcal{O}_1})} R^{K_{\mathcal{O}_2}+1}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2}\big(\mu\tau_{l_1}^{-2}\big) \;+\; I^{\square\,k_1,l_1}_{\mathcal{O}_1}\big(\mu\tau_{l_1}^{-2}\big)\Bigg]\\
&+ \sum_{\substack{\mathcal{O}_1\in\aleph^-(M)\\ \mathcal{O}_2\in\aleph^-(M_{\mathcal{O}_1})}} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_1}+\dim \mathfrak{m}_{\mathcal{O}_2}} \sum_{\substack{0\le k_1\le K_{\mathcal{O}_1}\\ 0\le k_2\le K_{\mathcal{O}_2}}}\sum_{\substack{l_1=0\\ l_2=0}}^{\infty} \frac{(i\mu)^{k_1+k_2}}{k_1!\,k_2!}\\
&\cdot \tau_{l_1}^{\dim W_{\mathcal{O}_1}-2(\dim \mathfrak{m}_{\mathcal{O}_2}+k_2)}\tau_{l_2}^{\dim W_{\mathcal{O}_2}} I^{k_1,l_1,k_2,l_2}_{\mathcal{O}_1,\mathcal{O}_2}\big(\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\big),
\end{aligned}$$

where the numbers $K_{\mathcal{O}_1}, K_{\mathcal{O}_2}, K_{\mathcal{O}_3} \in \{0,1,2\ldots\}$ are arbitrary. Passing on to the third iteration, we get

$$\begin{aligned}
I(\mu) = &\sum_{\mathcal{O}_1\in\aleph^+(M)} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_1}} \sum_{k_1=0}^{K_{\mathcal{O}_1}} \frac{(i\mu)^{k_1}}{k_1!}\mathcal{L}_{\mathcal{O}_1,k_1} \;+\; \sum_{\mathcal{O}_1\in\aleph(M)} R^{K_{\mathcal{O}_1}+1}_{\mathcal{O}_1}(\mu) \;+\; I^{\square}(\mu)\\
&+ \sum_{\mathcal{O}_1\in\aleph^-(M)} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_1}} \sum_{k_1=0}^{K_{\mathcal{O}_1}}\sum_{l_1=0}^{\infty} \frac{(i\mu)^{k_1}}{k_1!}\tau_{l_1}^{\dim W_{\mathcal{O}_1}} \Bigg[\sum_{\mathcal{O}_2\in\aleph^+(M_{\mathcal{O}_1})} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_2}}\\
&\cdot \sum_{k_2=0}^{K_{\mathcal{O}_2}} \frac{(i\mu)^{k_2}}{k_2!}\tau_{l_1}^{-2(\dim \mathfrak{m}_{\mathcal{O}_2}+k_2)}\mathcal{L}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2,k_2} \;+\; \sum_{\mathcal{O}_2\in\aleph(M_{\mathcal{O}_1})} R^{K_{\mathcal{O}_2}+1}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2}\big(\mu\tau_{l_1}^{-2}\big) \;+\; I^{\square\,k_1,l_1}_{\mathcal{O}_1}\big(\mu\tau_{l_1}^{-2}\big)\Bigg]\\
&+ \sum_{\substack{\mathcal{O}_1\in\aleph^-(M)\\ \mathcal{O}_2\in\aleph^-(M_{\mathcal{O}_1})}} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_1}+\dim \mathfrak{m}_{\mathcal{O}_2}} \sum_{\substack{0\le k_1\le K_{\mathcal{O}_1}\\ 0\le k_2\le K_{\mathcal{O}_2}}}\sum_{\substack{l_1=0\\ l_2=0}}^{\infty} \frac{(i\mu)^{k_1+k_2}}{k_1!\,k_2!}\tau_{l_1}^{\dim W_{\mathcal{O}_1}-2(\dim \mathfrak{m}_{\mathcal{O}_2}+k_2)}\tau_{l_2}^{\dim W_{\mathcal{O}_2}}\\
&\Bigg[\sum_{\mathcal{O}_3\in\aleph^+(M_{\mathcal{O}_2})} \Bigg[\big(2\pi\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\big)^{\dim \mathfrak{m}_{\mathcal{O}_3}} \sum_{k_3=0}^{K_{\mathcal{O}_3}} \frac{(i\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2})^{k_3}}{k_3!}\mathcal{L}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2,k_2,l_2,\mathcal{O}_3,k_3}\\
&+\; R^{K_{\mathcal{O}_3}+1}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2,k_2,l_2,\mathcal{O}_3}\big(\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\big)\Bigg]\\
&+ \sum_{\mathcal{O}_3\in\aleph^-(M_{\mathcal{O}_2})} \Bigg[\big(2\pi\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\big)^{\dim \mathfrak{m}_{\mathcal{O}_3}} \sum_{k_3=0}^{K_{\mathcal{O}_3}} \frac{(i\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2})^{k_3}}{k_3!}\sum_{l_3=0}^{\infty}\tau_{l_3}^{\dim W_{\mathcal{O}_3}} I^{k_1,l_1,k_2,l_2,k_3,l_3}_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3}\big(\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\tau_{l_3}^{-2}\big)\\
&+\; R^{K_{\mathcal{O}_3}+1}_{\mathcal{O}_1,k_1,l_1,\mathcal{O}_2,k_2,l_2,\mathcal{O}_3}\big(\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\big)\Bigg] \;+\; I^{\square\,k_1,l_1,k_2,l_2}_{\mathcal{O}_1,\mathcal{O}_2}\big(\mu\tau_{l_1}^{-2}\tau_{l_2}^{-2}\big)\Bigg],
\end{aligned}$$

where the numbers $K_{\mathcal{O}_1}, K_{\mathcal{O}_2}, K_{\mathcal{O}_3} \in \{0,1,2\ldots\}$ are arbitrary.

6.2. ***N*-th iteration.** To fix the problem that the lines corresponding to the "0-th iteration" look formally different to the other lines, let us write $M_{\mathcal{O}_0} := M$, $\mathfrak{g}_{\mathcal{O}_0} := \mathfrak{g}$, $\mathfrak{m}_{\mathcal{O}_0} := \{0\}$, $\nu_0 := \mu$, and define

the additional dummy notation

$$W_{\mathcal{O}_0} := M_{\mathcal{O}_0} = M, \quad V_{\mathcal{O}_0, G_{\mathcal{O}_0}} := \{0\}, \quad \aleph^-(M_{\mathcal{O}_{-1}}) := \{\mathcal{O}_0\}, \quad K_0 := 0, \quad L_N := \begin{cases} 1, & N = 0, \\ \infty, & N \in \mathbb{N}^*. \end{cases}$$

Furthermore, let us make the notational convention that an expression like $I_{\mathcal{O}_1,\dots,\mathcal{O}_0}$, where the last sub-sub-index is smaller than the first one, means that there is no sub-index, i.e. $I_{\mathcal{O}_1,\dots,\mathcal{O}_0} = I$, whereas $I_{\mathcal{O}_1,\dots,\mathcal{O}_1} = I_{\mathcal{O}_1}$, $I_{\mathcal{O}_1,\dots,\mathcal{O}_3} = I_{\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3}$, and so on. With this notation, we get for each $\Lambda \in \{1, 2, \dots\}$ and any choice of constants $K_{\mathcal{O}_1}, \dots, K_{\mathcal{O}_\Lambda} \in \{0, 1, 2, \dots\}$ the following formula for the result of the Λ-th iteration:

(6.2)

$$\begin{aligned} I(\mu) \;=\; & \sum_{N=0}^{\Lambda-1} \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0 \le j \le N}} (2\pi\mu)^{\sum_{j=0}^N \dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{\substack{0 \le k_j \le K_{\mathcal{O}_j},\\ 0 \le l_j < L_j,\\ 0 \le j \le N}} \frac{(i\mu)^{k_0+\dots+k_N}}{k_0! \cdots k_N!} \prod_{q=0}^N \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^N (\dim \mathfrak{m}_{\mathcal{O}_j} + k_j)} \\ & \cdot \Bigg[\sum_{\mathcal{O}_{N+1} \in \aleph^+(M_{\mathcal{O}_N})} (2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} \sum_{k_{N+1}=0}^{K_{\mathcal{O}_{N+1}}} \frac{(i\mu)^{k_{N+1}}}{k_{N+1}!} (\tau_{l_0} \cdots \tau_{l_N})^{-2(\dim \mathfrak{m}_{\mathcal{O}_{N+1}} + k_{N+1})} \\ & \cdot \mathcal{L}_{\mathcal{O}_1,k_1,l_1,\dots,\mathcal{O}_N,k_N,l_N,\mathcal{O}_{N+1},k_{N+1}} + \sum_{\mathcal{O}_{N+1} \in \aleph(M_{\mathcal{O}_N})} R^{K_{\mathcal{O}_{N+1}}+1}_{\mathcal{O}_1,k_1,l_1,\dots,\mathcal{O}_N,k_N,l_N,\mathcal{O}_{N+1}} \left(\mu \prod_{q=0}^N \tau_{l_q}^{-2} \right) \\ & + I^{\square\, k_1,l_1,\dots,k_N,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N} \left(\mu \prod_{q=0}^N \tau_{l_q}^{-2} \right) \Bigg] + \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0 \le j \le \Lambda}} (2\pi\mu)^{\sum_{j=0}^\Lambda \dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{\substack{0 \le k_j \le K_{\mathcal{O}_j},\\ 0 \le l_j < L_j,\\ 0 \le j \le \Lambda}} \frac{(i\mu)^{k_0+\dots+k_\Lambda}}{k_0! \cdots k_\Lambda!} \\ & \cdot \prod_{q=0}^\Lambda \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^\Lambda (\dim \mathfrak{m}_{\mathcal{O}_j} + k_j)} I^{k_1,l_1,\dots,k_\Lambda,l_\Lambda}_{\mathcal{O}_1,\dots,\mathcal{O}_\Lambda} \left(\mu \prod_{q=0}^\Lambda \tau_{l_q}^{-2} \right). \end{aligned}$$

Remark 6.1. In order to obtain a more compact expression in (6.2), we chose to collect the $\mathcal{O}_\bullet$, $k_\bullet$, and $l_\bullet$ summation indices under just two summation symbols. This has the drawback that the notation now hides almost all information about the summation order. While there is no confusion possible about the mutual order of the individual $\mathcal{O}_\bullet$-sums because each $\mathcal{O}_j$ is an element of an index set $\aleph(M_{\mathcal{O}_{j-1}})$ that depends on the previous summation index $\mathcal{O}_{j-1}$, the indices l_q for $q \ge 1$ are all summed over the same index set $\{0, 1, 2 \dots\}$, while the indices k_q are elements of index sets depending on $\mathcal{O}_{q-1}$ but not on k_{q-1} or any of the l-indices. Although the $k_\bullet$-sums are finite and we will later assume that the $\mathcal{O}_\bullet$-sums are finite as well (by the compact supports of all occurring amplitudes), the sums over the indices l_q are genuinely infinite for $q \ge 1$ and therefore the order of summation is crucial. As written down explicitly in the first lines of steps 2 and 3 of the iteration, the correct summation order is the following: For each $j \in \{1, 2, \dots, \}$, one first sums over the indices $l_j \in \{0, 1, 2, \dots\}$ (provided that they occur at all), then over the indices $k_j \in \{0, \dots, K_{\mathcal{O}_j}\}$, then over the orbits $\mathcal{O}_j \in \aleph(M_{\mathcal{O}_{j-1}})$, and then over the indices $l_{j-1} \in \{0, 1, 2, \dots\}$.

6.3. **End of iteration.** In the previous paragraph, we have seen that we can iterate the desingularization process as many times as we wish, obtaining in each interation step a precise formula for the oscillatory integral $I(\mu)$. We shall now see $-$ and this is the whole point of the iteration $-$ that after a certain number of iterations, all subsequent iterations become trivial in the sense that the summands they add to the inductive formula for $I(\mu)$ are equal to 0, so that the formula becomes stationary.

Consider an orbit $\mathcal{O}_1 \in \aleph(M_{\mathcal{O}_0})$ and an isotropy group $G_{\mathcal{O}_1}$ such that $\mathcal{O}_1 \cong G_{\mathcal{O}_0}/G_{\mathcal{O}_1}$. Since the open set $U_{\mathcal{O}_1}$ is pre-compact in $M_{\mathcal{O}_0}$, only finitely many $G_{\mathcal{O}_0}$-isotropy types occur in it. Denote the $G_{\mathcal{O}_0}$-isotropy types occurring in the intersection $U_{\mathcal{O}_1} \cap \Omega_{M_{\mathcal{O}_0}}$ by

$$\Big\{(H_1^{\mathcal{O}_1}), \dots, (H_{L_{\mathcal{O}_1}}^{\mathcal{O}_1})\Big\},$$

and suppose that they are indexed such that

$$(H_j^{\mathcal{O}_1}) \geq (H_{j'}^{\mathcal{O}_1}) \iff H_j^{\mathcal{O}_1} \text{ is conjugate to a subgroup of } H_{j'}^{\mathcal{O}_1} \implies j \geq j'.$$

In particular, by Proposition 4.12,

$$(H_1^{\mathcal{O}_1}) = (G_{\mathcal{O}_1}).$$

Now, recall from Proposition 4.14 that the regular isotropy types in all connected components of $U_{\mathcal{O}_1} \cap \Omega_{M_{\mathcal{O}_0}}$ agree. This unique regular isotropy type is equal to $(H^{\Omega}_{\mathrm{reg}})$ because $U_{\mathcal{O}_1} \cap \Omega_{M_{\mathcal{O}_0}}$ is an open subset of $\Omega_{M_{\mathcal{O}_0}}$. As we chose to partially order the isotropy types, we conclude that one has

$$\big(H_{L_{\mathcal{O}_1}}^{\mathcal{O}_1}\big) = \big(H^{\Omega}_{\mathrm{reg}}\big).$$

Now, suppose that in fact we have $\mathcal{O}_1 \in \aleph^-(M_{\mathcal{O}_0})$, which implies that $L_{\mathcal{O}_1} \geq 2$, and we can pass to the second iteration of the desingularization process. Then, by Proposition 5.2, we know that the set of isotropy types occurring in $\Omega_{M_{\mathcal{O}_1}}$ is precisely given by

$$\Big\{\big(H_2^{\mathcal{O}_1}\big), \dots, \big(H^{\Omega}_{\mathrm{reg}}\big)\Big\}, \tag{6.3}$$

and, moreover, $\big(H^{\Omega}_{\mathrm{reg}}\big)$ is the regular isotropy type in all connected components of $\Omega_{M_{\mathcal{O}_1}}$. Consider now an orbit $\mathcal{O}_2 \in \aleph(M_{\mathcal{O}_1})$ and an isotropy group $G_{\mathcal{O}_2}$ such that $\mathcal{O}_2 \cong G_{\mathcal{O}_1}/G_{\mathcal{O}_2}$. We necessarily have

$$(G_{\mathcal{O}_2}) = (H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}}) \qquad \text{for some } j_{\mathcal{O}_2} \in \{2, \dots, L_{\mathcal{O}_1}\},$$

and, by Corollary 4.13, the set of $G_{\mathcal{O}_1}$-isotropy types occurring in $U_{\mathcal{O}_2} \cap \Omega_{M_{\mathcal{O}_1}}$ is a subset of the set (6.3) that contains $(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}})$ and $\big(H^{\Omega}_{\mathrm{reg}}\big)$. In particular, $\big(H^{\Omega}_{\mathrm{reg}}\big)$ is the regular isotropy type in all connected components of $U_{\mathcal{O}_2} \cap \Omega_{M_{\mathcal{O}_1}}$. Consequently, if $\mathcal{O}_2 \in \aleph^-(M_{\mathcal{O}_1})$, then, by Proposition 4.14, the $G_{\mathcal{O}_2}$-isotropy types occurring in $\Omega_{M_{\mathcal{O}_2}}$ or $\aleph(M_{\mathcal{O}_2})$ form a subset of

$$\Big\{\big(H_2^{\mathcal{O}_1}\big), \dots, \big(H^{\Omega}_{\mathrm{reg}}\big)\Big\} - \big\{(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}})\big\} \tag{6.4}$$

consisting of isotropy types that are strictly greater than $(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}})$. Moreover, the regular isotropy type in any of the connected components of $\Omega_{M_{\mathcal{O}_2}}$ is equal to $\big(H^{\Omega}_{\mathrm{reg}}\big)$. Repeating these arguments for the N-th time yields for any tuple $(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N)$ with $\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}})$, $j \in \{1, \dots, N\}$, that the $G_{\mathcal{O}_N}$-isotropy types occurring in $\Omega_{M_{\mathcal{O}_N}}$ form a subset of

$$\Big\{(H_2^{\mathcal{O}_1}), \dots, (H^{\Omega}_{\mathrm{reg}})\big\} - \big\{(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}}), \dots, (H^{\mathcal{O}_1}_{j_{\mathcal{O}_N}})\big\} \tag{6.5}$$

consisting of isotropy types that are strictly greater than $(H^{\mathcal{O}_1}_{j_{\mathcal{O}_N}})$, while $(H^{\Omega}_{\mathrm{reg}})$ is the regular isotropy type in any of the connected components of $\Omega_{M_{\mathcal{O}_N}}$. In particular, the set $\big\{(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}}), \dots, (H^{\mathcal{O}_1}_{j_{\mathcal{O}_N}})\big\}$ possesses a total ordering given by

$$(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}}) < \cdots < (H^{\mathcal{O}_1}_{j_{\mathcal{O}_N}}).$$

Thus, as soon as N is large enough that the set $\Big\{\big(H_1^{\mathcal{O}_1}\big), \dots, \big(H_{L_{\mathcal{O}_1}}^{\mathcal{O}_1}\big) = \big(H^{\Omega}_{\mathrm{reg}}\big)\Big\}$ does not contain any totally ordered subset of length $N+1$, then

$$\Big\{(H_2^{\mathcal{O}_1}), \dots, (H^{\Omega}_{\mathrm{reg}})\big\} - \big\{(H^{\mathcal{O}_1}_{j_{\mathcal{O}_2}}), \dots, (H^{\mathcal{O}_1}_{j_{\mathcal{O}_N}})\big\} = \{(H^{\Omega}_{\mathrm{reg}})\}.$$

If we assume now that $(H^{\Omega}_{\mathrm{reg}})$ is the principal isotropy type in $U_{\mathcal{O}}$ (and not just the regular isotropy type in $U_{\mathcal{O}} \cap \Omega_{M_{\mathcal{O}_0}}$), the iteration stops: By Proposition 5.2, one has $W_{\mathcal{O}_{N+1}} = \{0\}$ for all $\mathcal{O}_{N+1} \subset \Omega_{M_{\mathcal{O}_N}}$, which implies $\aleph^-(M_{\mathcal{O}_N}) = \emptyset$. Similarly, if we assume that $\dim H^{\Omega}_{\mathrm{reg}} = 0$, the iteration stops because

$\mathfrak{g}_{\mathcal{O}_{N+1}} = \{0\}$ for all $\mathcal{O}_{N+1} \subset \Omega_{M_{\mathcal{O}_N}}$, which implies $\aleph^-(M_{\mathcal{O}_N}) = \emptyset$. We can summarize our findings in the following

Theorem 6.2 (**End of iteration).** *Let $\mathcal{O}_1 \in \aleph(M_{\mathcal{O}_0})$ be an orbit, and let $\Lambda_{\mathcal{O}_1} \in \mathbb{N}$ be the maximal number of a totally ordered subset of the set of isotropy types occurring in $U_{\mathcal{O}_1} \cap \Omega_{M_{\mathcal{O}_0}}$. Assume either that $(H^\Omega_{\mathrm{reg}})$ is the principal isotropy type in $U_{\mathcal{O}}$ or that $\dim H^\Omega_{\mathrm{reg}} = 0$. Then, the iterated desingularization process applied to $U_{\mathcal{O}_1}$ stops in* Case 1 *after at most $\Lambda_{\mathcal{O}_1} + 1$ iterations. Moreover, the regular isotropy types in all connected components of the zero level sets of the momentum maps in any iteration of the desingularization process agree and correspond to the unique regular $G_{\mathcal{O}_0}$-isotropy type $(H^\Omega_{\mathrm{reg}})$ in $\Omega_{M_{\mathcal{O}_0}}$. In the manifolds occurring in the last non-trivial iteration step, all orbits in the zero level sets of the momentum maps are of regular type $(H^\Omega_{\mathrm{reg}})$.* □

To obtain a global statement, we now use that the support of the amplitude a in the oscillatory integral $I(\mu)$ is compact and put

$$\Lambda(a) := \max\{\Lambda_{\mathcal{O}_1} : \mathcal{O}_1 \in \aleph(M_{\mathcal{O}_0}),\ U_{\mathcal{O}_1} \cap (\mathrm{pr}_M(\mathrm{supp}\ a)) \neq \emptyset\},$$

where the numbers $\Lambda_{\mathcal{O}_1}$ are defined as in Theorem 6.2. Assuming from now on that either $(H^\Omega_{\mathrm{reg}})$ is the principal isotropy type in M or that $\dim H^\Omega_{\mathrm{reg}} = 0$, the formula for $I(\mu)$ then becomes

(6.6)

$$\begin{aligned} I(\mu) = &\sum_{N=0}^{\Lambda(a)} \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0\leq j\leq N}} (2\pi\mu)^{\sum_{j=0}^N \dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{\substack{0\leq k_j \leq K_{\mathcal{O}_j},\\ 0\leq l_j < L_j,\\ 0\leq j\leq N}} \frac{(i\mu)^{k_0+\ldots+k_N}}{k_0!\cdots k_N!} \prod_{q=0}^N \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^N (\dim \mathfrak{m}_{\mathcal{O}_j} + k_j)} \\ &\cdot \Bigg[\sum_{\mathcal{O}_{N+1} \in \aleph^+(M_{\mathcal{O}_N})} \Bigg((2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} \sum_{k_{N+1}=0}^{K_{\mathcal{O}_{N+1}}} \frac{(i\mu)^{k_{N+1}}}{k_{N+1}!} (\tau_{l_0}\cdots\tau_{l_N})^{-2(\dim \mathfrak{m}_{\mathcal{O}_{N+1}} + k_{N+1})} \\ &\cdot \mathcal{L}_{\mathcal{O}_1,k_1,l_1,\ldots,\mathcal{O}_N,k_N,l_N,\mathcal{O}_{N+1},k_{N+1}} \\ &+\ R^{K_{\mathcal{O}_{N+1}}+1}_{\mathcal{O}_1,k_1,l_1,\ldots,\mathcal{O}_N,k_N,l_N,\mathcal{O}_{N+1}} \left(\mu \prod_{q=0}^N \tau_{l_q}^{-2}\right)\Bigg) \ +\ I^{\square\, k_1,l_1,\ldots,k_N,l_N}_{\mathcal{O}_1,\ldots,\mathcal{O}_N} \left(\mu \prod_{q=0}^N \tau_{l_q}^{-2}\right)\Bigg]. \end{aligned}$$

The numbers $K_{\mathcal{O}_1}, \ldots, K_{\mathcal{O}_{\Lambda(a)}} \in \{0,1,2,\ldots\}$ here are arbitrary. Note that the coefficients

$$\mathcal{L}_{\mathcal{O}_1,k_1,l_1,\ldots,\mathcal{O}_N,k_N,l_N,\mathcal{O}_{N+1},k_{N+1}}$$

on the right hand side of (6.6) are explicitly computable in terms of the amplitude a and independent of μ, while the summands in the subsequent lines represent remainder terms. Remarkably, there are no terms left to which we would need to apply the stationary phase theorem!

In principle, the formula (6.6) represents a full asymptotic expansions of $I(\mu)$ up to arbitrary order in μ. However, re-writing the higher order terms in a useful form is much more difficult than the same task for the first order leading term. In this paper, we shall be satisfied with computing only a first order asymptotic formula for $I(\mu)$, therefore we simplify the following discussion by choosing $K_{\mathcal{O}_1} = \ldots = K_{\mathcal{O}_N} = 0$ for all tuples $(\mathcal{O}_1, \ldots, \mathcal{O}_N)$, $1 \leq N \leq \Lambda(a)$. This simplifies formula (6.6)

considerably to

$$I(\mu) = \sum_{N=0}^{\Lambda(a)} \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0\leq j\leq N}} (2\pi\mu)^{\sum_{j=0}^N \dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{\substack{0\leq l_j<L_j,\\ 0\leq j\leq N}} \prod_{q=0}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^N \dim \mathfrak{m}_{\mathcal{O}_j}}$$

$$\Bigg[\sum_{\mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})} \Bigg((2\pi\mu)^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} (\tau_{l_0}\cdots\tau_{l_N})^{-2\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}$$

$$+ R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}} \left(\mu \prod_{q=0}^N \tau_{l_q}^{-2}\right)\Bigg) + I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N} \left(\mu \prod_{q=0}^N \tau_{l_q}^{-2}\right)\Bigg] \tag{6.7}$$

We point out again that the infinite sums occurring here are convergent by construction provided that the $l_\bullet$- and $\mathcal{O}_\bullet$-sums are carried out in the correct order, see Remark 6.1.

7. Computation of the leading term and remainder estimates

Under the assumption that $(H^\Omega_{\mathrm{reg}})$ is either the principal isotropy type in M or $\dim H^\Omega_{\mathrm{reg}} = 0$, all that is left to do in order to understand the first-order asymptotic behavior of the oscillatory integral $I(\mu)$ as $\mu \to 0$ is expressing the right hand side of (6.7) properly in terms of the original amplitude a, and separating the terms into a leading term and a remainder term.

In essence, the computation of the leading term will be accomplished by following the steps in the various iterations *backwards*. An important aspect of this reversed iteration is that the sums occurring in (6.7), which were introduced somewhat artificially in the iterated desingularization process, are each evaluated to 1 in the order as they occur, in particular convergence issues are naturally avoided in the computation of the leading term.

7.1. **Computation of the leading term.** The canonical candidates for the summands that make up the leading term in (6.7) are the summands involving the coefficients $\mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}$, which are themselves leading terms of asymptotic expansions carried out in the various iteration steps. In order to show that the sum

$$L_0 (2\pi\mu)^{\dim G_{\mathcal{O}_0} - \dim H^\Omega_{\mathrm{reg}}}$$

$$:= \sum_{N=0}^{\Lambda(a)} \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0\leq j\leq N,\\ \mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})}} (2\pi\mu)^{\sum_{j=0}^{N+1} \dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{l_j=0}^{\infty} \prod_{q=0}^{N+1} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N+1} \dim \mathfrak{m}_{\mathcal{O}_j}} \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}$$

of all the $\mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}$ as occurring in (6.7) is finite, and to express it as explicitly as possible in terms of the original amplitude a, we just need to undo all the steps that were used to define these coefficients inductively. We will do this separately for each maximal orbit tuple $(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})$, which means that we follow the iterated desingularization process backwards and assume that the $(N+1)$-th iteration is the last non-trivial one. By Theorem 6.2, we then have

$$\aleph(M_{\mathcal{O}_N}) = \aleph^+(M_{\mathcal{O}_N}), \qquad (G_{\mathcal{O}_{N+1}}) = (H^\Omega_{\mathrm{reg}}),$$

and (5.27) yields

$$\mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0} = \frac{\operatorname{vol} G_{\mathcal{O}_N}}{\operatorname{vol} H^\Omega_{\mathrm{reg}}} \int_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})} \fint_{\mathcal{O}'_{N+1}} \int_{\mathfrak{g}_{\mathcal{O}_N\, p}} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\, dX$$
$$\chi_{U_{\mathcal{O}_{N+1}}}(p)\, d\mathcal{O}'_{N+1}(p)\, d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1}). \tag{7.1}$$

Note that the only way in which the right hand side above depends on $\mathcal{O}_{N+1}$ is via the cutoff function $\chi_{U_{\mathcal{O}_{N+1}}}$. Keeping $(\mathcal{O}_0,\dots,\mathcal{O}_N)$ fixed, we now sum over all orbits $\mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})=\aleph(M_{\mathcal{O}_N})$. Since

$$\sum_{\mathcal{O}_{N+1}\in\aleph(M_{\mathcal{O}_N})}\chi_{U_{\mathcal{O}_{N+1}}}=1,$$

the sum vanishes and we get

$$\sum_{\mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})}\mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}$$
$$=\frac{\operatorname{vol}G_{\mathcal{O}_N}}{\operatorname{vol}H^\Omega_{\mathrm{reg}}}\int\limits_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})}\;\fint\limits_{\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\,dX\,d\mathcal{O}'_{N+1}(p)\,d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1}).$$

The reduced amplitudes occurring here are defined by

$$a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)=a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\tau_{l_N}\,p,X)\chi_{M_{\mathcal{O}}}(p),\qquad p\in M_{\mathcal{O}_N}\subset W_{\mathcal{O}_N},\ X\in\mathfrak{g}_{\mathcal{O}_N},$$

so that one has

$$\begin{aligned}\fint\limits_{\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\,dX\,d\mathcal{O}'_{N+1}(p)&=\int\limits_{\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\tau_{l_N}\,p,X)\chi_{M_{\mathcal{O}}}(p)\,dX\,d\mathcal{O}'_{N+1}(p)\\&=\fint\limits_{\tau_{l_N}\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N\,\tau^{-1}_{l_N}p}}a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{M_{\mathcal{O}}}(\tau^{-1}_{l_N}p)\,dX\,d(\tau_{l_N}\mathcal{O}'_{N+1})(p)\\&=\fint\limits_{\tau_{l_N}\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{\mathscr{W}^l_{\mathcal{O}}}(p)\,dX\,d(\tau_{l_N}\mathcal{O}'_{N+1})(p),\end{aligned}$$

where we used that $\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N}$ is locally around an orbit $\mathcal{O}'_{N+1}$ symplectomorphic to an open subset of the vector space $V_{\mathcal{O}'_{N+1},\mathcal{O}'_{N+1}}$, the scalar multiplication of $\mathcal{O}'_{N+1}$ with τ_{l_N} is to be understood as being carried out in $V_{\mathcal{O}'_{N+1},\mathcal{O}'_{N+1}}$, the scalar multiplication of p with $\tau^{-1}_{l_N}$ is to be understood as being carried out in $W_{\mathcal{O}_N}$, and we also used that one has $\mathfrak{g}_{\mathcal{O}_N p}=\mathfrak{g}_{\mathcal{O}_N\lambda p}$ for any $\lambda\in\mathbb{R}-\{0\}$, since the $G_{\mathcal{O}_N}$-action on $W_{\mathcal{O}_N}$ is linear. We arrive at

$$\begin{aligned}&\int\limits_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})}\;\fint\limits_{\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\,dX\,d\mathcal{O}'_{N+1}(p)\,d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1})\\&=\int\limits_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})}\;\fint\limits_{\tau_{l_N}\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{\mathscr{W}^{l_N}_{\mathcal{O}}}(p)\,dX\,d(\tau_{l_N}\mathcal{O}'_{N+1})(p)\\&\qquad\qquad d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1})\\&=\tau_{l_N}^{-\dim\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N}}\int\limits_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})}\;\fint\limits_{\mathcal{O}'_{N+1}}\int\limits_{\mathfrak{g}_{\mathcal{O}_N p}}a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{\mathscr{W}^{l_N}_{\mathcal{O}}}(p)\,dX\,d(\mathcal{O}'_{N+1})(p)\\&\qquad\qquad d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1}).\end{aligned}$$

Next, we recall from (4.30) and (4.31) that one has

$$\dim W_{\mathcal{O}_N}-2\dim\mathfrak{m}_{\mathcal{O}_{N+1}}-\dim\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N}=\dim W_{\mathcal{O}_{N+1}}=0\qquad\forall\,\mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N}),$$

so that when summing in the correct order (compare Remark 6.1) we obtain

(7.2)

$$\sum_{l_N=0}^{\infty} \tau_{l_N}^{\dim W_{\mathcal{O}_N}-2\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} \sum_{\mathcal{O}_{N+1}\in \aleph^+(M_{\mathcal{O}_N})} \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}$$

$$= \frac{\operatorname{vol} G_{\mathcal{O}_N}}{\operatorname{vol} H^{\Omega}_{\text{reg}}} \sum_{l_N=0}^{\infty} \tau_{l_N}^{\dim W_{\mathcal{O}_N}-2\dim \mathfrak{m}_{\mathcal{O}_{N+1}}-\dim \Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N}}$$

$$\int_{(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N})} \fint_{\mathcal{O}'_{N+1}} \int_{\mathfrak{g}_{\mathcal{O}_N p}} a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{\mathscr{W}^{l_N}_{\mathcal{O}}}(p)\, dX\, d(\mathcal{O}'_{N+1})(p)\, d(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1})$$

$$= \frac{\operatorname{vol} G_{\mathcal{O}_N}}{\operatorname{vol} H^{\Omega}_{\text{reg}}} \int_{(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N})} \fint_{\mathcal{O}'_{N+1}} \int_{\mathfrak{g}_{\mathcal{O}_N p}} a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X) \underbrace{\Big(\sum_{l_N=0}^{\infty} \chi_{\mathscr{W}^{l_N}_{\mathcal{O}}}(p)\Big)}_{=1} dX\, d(\mathcal{O}'_{N+1})(p)$$
$$d(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1})$$

$$= \frac{\operatorname{vol} G_{\mathcal{O}_N}}{\operatorname{vol} H^{\Omega}_{\text{reg}}} \int_{(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N})} \fint_{\mathcal{O}'_{N+1}} \int_{\mathfrak{g}_{\mathcal{O}_N p}} a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\, dX\, d(\mathcal{O}'_{N+1})(p)$$
$$d(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1}).$$

This formula represents the result of the backwards step from the $(N+1)$-th to the N-th iteration. To go further back to the $(N-1)$-th iteration, we consider the orbits

$$\mathcal{O}'_{N+1} \subset \Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})} \subset \Phi^{-1}_{W_{\mathcal{O}_N}}(0)_{(H^{\Omega}_{\text{reg}})} \subset W_{\mathcal{O}_N}$$

as subsets of $W_{\mathcal{O}_N}$, which we indicate by writing w instead of p for a point in an orbit $\mathcal{O}'_{N+1}$, and we recall from (5.9) and (5.22) that $a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}$ was defined as

$$(7.3)\quad a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(w,X) = a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(0,w,X) = \int_{G_{\mathcal{O}_{N-1}}} \int_{V_{\mathcal{O}_N,G_{\mathcal{O}_N}}} (\beta(\pi(g,0,v,w)))^{-1}$$
$$a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,v,w)), \operatorname{Ad}(g)^{-1}(X)\big)\chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,v,w)))\, dv\, dg, \qquad (w,X)\in W_{\mathcal{O}_N}\times \mathfrak{g}_{\mathcal{O}_N}.$$

From (4.38) and (4.46), it follows that there is a symplectomorphism

$$(7.4)\qquad \alpha_{\mathcal{O}_N} : V_{\mathcal{O}_N,G_{\mathcal{O}_N}} \times \big(\Phi^{-1}_{W_{\mathcal{O}_N}}(0)_{(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N}\big) \xrightarrow{\cong} \mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_{N-1}}.$$

Furthermore, one has the equality of open sets in $\Phi^{-1}_{W_{\mathcal{O}_N}}(0)_{(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N}$

$$(7.5)\qquad \Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\text{reg}})}/G_{\mathcal{O}_N} = (\Phi^{-1}_{W_{\mathcal{O}_N}}(0)_{(H^{\Omega}_{\text{reg}})} \cap M_{\mathcal{O}_N})/G_{\mathcal{O}_N}.$$

We can now use (7.3) and then pull back the resulting integral via (7.4) after changing the order of integration:

$$\begin{aligned}
\Gamma &:= \int_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})} \fint_{\mathcal{O}'_{N+1}} \int_{\mathfrak{g}_{\mathcal{O}_N p}} a^{0,l_1,\dots,l_{N-1},0}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\, dX\, d(\mathcal{O}'_{N+1})(p)\, d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1})\\
&= \int_{V_{\mathcal{O}_N,G_{\mathcal{O}_N}}} \int_{(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})} \fint_{\mathcal{O}'_{N+1}} \int_{\mathfrak{g}_{\mathcal{O}_N w}} \int_{G_{\mathcal{O}_{N-1}}}\\
&\qquad (\beta(\pi(g,0,v,w)))^{-1} a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,v,w)), \mathrm{Ad}\,(g)^{-1}(X)\big)\chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,v,w)))\\
&\qquad\qquad dg\, dX\, d(\mathcal{O}'_{N+1})(w)\, d(\Omega_{M_{\mathcal{O}_N}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1})\, dv\\
&= \int_{\mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} \int_{G_{\mathcal{O}_{N-1}}} (\mathrm{vol}\, G_{\mathcal{O}_N})^{-1} \int_{G_{\mathcal{O}_N}} \int_{(\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}\cdot h}}\\
&\quad (\beta(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N}\cdot h)))^{-1} a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N}\cdot h)), \mathrm{Ad}\,(g)^{-1}(X)\big)\\
&\qquad \chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N}\cdot h)))\, dX\, dh\, dg\, d\big(\mathcal{J}^{-1}_{\mathcal{O}_{N-1}}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}\big)(\mathcal{O}'_N).
\end{aligned}$$

Here, we have changed the order of integration again in the last step, $p_{\mathcal{O}'_N} \in V_{\mathcal{O}_N,G_{\mathcal{O}_N}}$ is the unique point such that

$$\mathrm{pr}_{V_{\mathcal{O}_N,G_{\mathcal{O}_N}}}(\mathcal{O}'_N) = \{p_{\mathcal{O}'_N}\},$$

and $w_{\mathcal{O}'_N} \in \mathrm{pr}_{W_{\mathcal{O}_N}}(\mathcal{O}'_N)$ is an arbitrary point. Now, we perform in the dg-integral the substitution $g := hg'$, which leaves the Haar measure dg invariant. Then, since

$$\pi(hg,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N}\cdot h) = \pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N}), \qquad (\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}\cdot h} = \mathrm{Ad}\,(h)\big((\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}}\big),$$

we obtain

$$\begin{aligned}
\Gamma &= \int_{\mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} \int_{G_{\mathcal{O}_{N-1}}} (\mathrm{vol}\, G_{\mathcal{O}_N})^{-1} \int_{G_{\mathcal{O}_N}} \int_{\mathrm{Ad}\,(h)((\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}})}\\
&\quad (\beta(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})))^{-1} a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})), \mathrm{Ad}\,(g)^{-1}\mathrm{Ad}\,(h)^{-1}(X)\big)\\
&\qquad \chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})))\, dX\, dh\, dg\, d\big(\mathcal{J}^{-1}_{\mathcal{O}_{N-1}}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}\big)(\mathcal{O}'_N).
\end{aligned}$$

We proceed by performing in the dX-integral the substitution $X := \mathrm{Ad}\,(h)(X')$, which leaves the measure dX invariant because it was defined using an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$. This yields

$$\begin{aligned}
\Gamma &= \int_{\mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} \int_{G_{\mathcal{O}_{N-1}}} (\mathrm{vol}\, G_{\mathcal{O}_N})^{-1} \int_{G_{\mathcal{O}_N}} \int_{(\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}}}\\
&\quad (\beta(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})))^{-1} a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})), \mathrm{Ad}\,(g)^{-1}(X)\big)\\
&\qquad \chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})))\, dX\, dh\, dg\, d\big(\mathcal{J}^{-1}_{\mathcal{O}_{N-1}}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}\big)(\mathcal{O}'_N).
\end{aligned}$$

The integrand is now $G_{\mathcal{O}_N}$-invariant, so that the normalization over $G_{\mathcal{O}_N}$ evaluates trivially to 1:

$$\begin{aligned}
\Gamma &= \int_{\mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} \int_{G_{\mathcal{O}_{N-1}}} \int_{(\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}}}\\
&\quad (\beta(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})))^{-1} a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})), \mathrm{Ad}\,(g)^{-1}(X)\big)\\
&\qquad \chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(\pi(g,0,p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})))\, dX\, dg\, d\big(\mathcal{J}^{-1}_{\mathcal{O}_{N-1}}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}\big)(\mathcal{O}'_N).
\end{aligned}$$

To proceed, we write

$$y_{\mathcal{O}_N} := \pi(e, 0, p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})$$

and recall from (4.44) that one has

$$(\mathfrak{g}_{\mathcal{O}_N})_{w_{\mathcal{O}'_N}} = (\mathfrak{g}_{\mathcal{O}_{N-1}})_{\varphi^{-1}_{\mathcal{O}_N}(\pi(e,0,p_{\mathcal{O}'_N},w_{\mathcal{O}'_N}))} = (\mathfrak{g}_{\mathcal{O}_{N-1}})_{\varphi^{-1}_{\mathcal{O}_N}(y_{\mathcal{O}_N})}.$$

Furthermore, (5.19) says

$$\beta(\pi(g, 0, p_{\mathcal{O}'_N}, w_{\mathcal{O}'_N})) = \operatorname{vol} G_{\mathcal{O}_N}. \tag{7.6}$$

Using also the relation

$$(\mathfrak{g}_{\mathcal{O}_{N-1}})_{g\cdot\varphi^{-1}_{\mathcal{O}_N}(y_{\mathcal{O}_N})} = (\mathfrak{g}_{\mathcal{O}_{N-1}})_{\varphi^{-1}_{\mathcal{O}_N}(g\cdot y_{\mathcal{O}_N})} = \mathrm{Ad}\,(g)^{-1}((\mathfrak{g}_{\mathcal{O}_{N-1}})_{\varphi^{-1}_{\mathcal{O}_N}(y_{\mathcal{O}_N})}),$$

we arrive at

$$\begin{aligned}
\Gamma &= (\operatorname{vol} G_{\mathcal{O}_N})^{-1} \int_{\mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} \int_{G_{\mathcal{O}_{N-1}}} \int_{(\mathfrak{g}_{\mathcal{O}_{N-1}})_{g\cdot\varphi^{-1}_{\mathcal{O}_N}(y_{\mathcal{O}_N})}} \\
&\quad a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}\big(\varphi^{-1}_{\mathcal{O}_N}(g\cdot y_{\mathcal{O}_N}), X\big)\chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(g\cdot y_{\mathcal{O}_N}))\, dX\, dg\, d\big(\mathcal{J}^{-1}_{\mathcal{O}_{N-1}}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}\big)(\mathcal{O}'_N) \\
&= \frac{\operatorname{vol} G_{\mathcal{O}_{N-1}}}{\operatorname{vol} G_{\mathcal{O}_N}} \int_{\mathcal{J}^{-1}_{\mathcal{O}_N}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} ⨍_{\mathcal{O}'_N} \int_{(\mathfrak{g}_{\mathcal{O}_{N-1}})_{\varphi^{-1}_{\mathcal{O}_N}(y)}} \\
&\quad a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}(\varphi^{-1}_{\mathcal{O}_N}(y), X)\chi_{U_{\mathcal{O}_N}}(\varphi^{-1}_{\mathcal{O}_N}(y))\, dX\, d\mathcal{O}'_N(y)\, d\big(\mathcal{J}^{-1}_{\mathcal{O}_{N-1}}(0)_{(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}\big)(\mathcal{O}'_N).
\end{aligned}$$

Pulling back the integral along $\widetilde{\varphi}_{\mathcal{O}_N}$ using (4.29) yields

$$\begin{aligned}
\Gamma = \frac{\operatorname{vol} G_{N-1}}{\operatorname{vol} G_{\mathcal{O}_N}} \int_{\Omega_{M_{\mathcal{O}_{N-1}}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} ⨍_{\mathcal{O}_N} \int_{(\mathfrak{g}_{\mathcal{O}_{N-1}})_p} & a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}(p, X)\, dX \\
& \chi_{U_{\mathcal{O}_N}}(p)\, d\mathcal{O}'_N(p)\, d(\Omega_{M_{\mathcal{O}_{N-1}}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}})(\mathcal{O}'_N),
\end{aligned} \tag{7.7}$$

similarly as in our earlier computations leading to the formula (5.27). Inserting the result (7.7) into (7.2) gives us the formula

$$\begin{aligned}
&\sum_{l_N=0}^{\infty} \tau_{l_N}^{\dim W_{\mathcal{O}_N} - 2\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} \sum_{\mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})} \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0} \\
&= \frac{\operatorname{vol} G_{\mathcal{O}_{N-1}}}{\operatorname{vol} H^\Omega_{\mathrm{reg}}} \int_{\Omega_{M_{\mathcal{O}_{N-1}}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}}} ⨍_{\mathcal{O}'_N} \int_{(\mathfrak{g}_{\mathcal{O}_{N-1}})_p} a^{0,l_1,\dots,0,l_{N-1}}_{\mathcal{O}_1,\dots,\mathcal{O}_{N-1}}(p, X)\, dX\, \chi_{U_{\mathcal{O}_N}}(p)\, d\mathcal{O}'_N(p) \\
&\qquad\qquad d(\Omega_{M_{\mathcal{O}_{N-1}}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_{N-1}})(\mathcal{O}'_N) \\
&= \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N},
\end{aligned} \tag{7.8}$$

where the final equality is precisely (7.1) applied to N instead of $N+1$. Using the result (7.8), we can now repeat the reversed iteration until $N=0$. To state the result after having iterated back to $N=0$, note that one has for every maximal tuple $(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})$ the relation

$$\begin{aligned}
\sum_{j=0}^{N+1} \dim \mathfrak{m}_{\mathcal{O}_j} &= \sum_{j=0}^{N} \dim \mathfrak{g}_{\mathcal{O}_j} - \dim \mathfrak{g}_{\mathcal{O}_{j+1}} + \dim \mathfrak{g}_{\mathcal{O}_{N+1}} = \dim \mathfrak{g}_{\mathcal{O}_0} - \dim \mathfrak{g}_{\mathcal{O}_{N+1}} \\
&= \dim G_{\mathcal{O}_0} - \dim H^\Omega_{\mathrm{reg}}.
\end{aligned} \tag{7.9}$$

Thus, we arrive at

Proposition 7.1. *The leading term L_0 introduced at the beginning of Section 7.1 is given by*

$$L_0 = \frac{\operatorname{vol} G_{\mathcal{O}_0}}{\operatorname{vol} H^{\Omega}_{\mathrm{reg}}} \int\limits_{\Omega_{M_{\mathcal{O}_0}(H^{\Omega}_{\mathrm{reg}})}/G_{\mathcal{O}_0}} \fint\limits_{\mathcal{O}'_1} \int\limits_{(\mathfrak{g}_{\mathcal{O}_0})_p} a(p,X)\, dX\, d\mathcal{O}'_1(p)\, d(\Omega_{M_{\mathcal{O}_0}(H^{\Omega}_{\mathrm{reg}})}/G_{\mathcal{O}_0})(\mathcal{O}'_1). \tag{7.10}$$

Proof. An iterative application of (7.8) yields

$$\begin{aligned}
&\sum_{N=0}^{\Lambda(a)} \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0\le j\le N,\\ \mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})}} (2\pi\mu)^{\sum_{j=0}^{N+1}\dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{l_j=0}^{\infty} \prod_{q=0}^{N+1} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N+1}\dim\mathfrak{m}_{\mathcal{O}_j}} \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}\\
&= \frac{\operatorname{vol} G_{\mathcal{O}_0}}{\operatorname{vol} H^{\Omega}_{\mathrm{reg}}} (2\pi\mu)^{\dim G_{\mathcal{O}_0} - \dim H^{\Omega}_{\mathrm{reg}}} \int\limits_{\Omega_{M_{\mathcal{O}_0}(H^{\Omega}_{\mathrm{red}})/G_{\mathcal{O}_0}}} \fint\limits_{\mathcal{O}'_1} \int\limits_{(\mathfrak{g}_{\mathcal{O}_0})_p} a(p,X)\, dX\, d\mathcal{O}'_1(p)\, d(\Omega_{M_{\mathcal{O}_0}(H^{\Omega}_{\mathrm{red}})}/G_{\mathcal{O}_0})(\mathcal{O}'_1),
\end{aligned}$$

and the assertion follows. Note that the leading term is completely independent of the constructions and choices involved in the desingularization process. □

7.2. **Remainder estimate I.** We are now going to estimate the contributions in formula (6.7) coming from those summands that arise as remainder terms in the stationary phase approximations that are carried out in each *Case 1*-iteration of the desingularization process. To begin with the estimates, fix a maximal orbit tuple $(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})$ (which means that $W_{\mathcal{O}_{N+1}} = \{0\}$ and $(G_{\mathcal{O}_{N+1}}) = (H^{\Omega}_{\mathrm{reg}})$) and consider the remainder term

$$\begin{aligned}
R^{\text{Case 1}}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N) &= I^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}(\nu_N) - (2\pi\nu_N)^{\dim\mathfrak{m}_{\mathcal{O}_{N+1}}} \mathcal{L}_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1},0}\\
&= \int\limits_{\mathfrak{g}_{\mathcal{O}_N}} \int\limits_{U_{\mathcal{O}_{N+1}}} e^{i\psi_{M_{\mathcal{O}_N}}(p,X)/\nu_N} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{U_{\mathcal{O}_{N+1}}}(p)\, dp\, dX\\
&\quad - (2\pi\nu_N)^{\dim\mathfrak{m}_{\mathcal{O}_{N+1}}} \frac{\operatorname{vol} G_{\mathcal{O}_N}}{\operatorname{vol} H^{\Omega}_{\mathrm{reg}}} \int\limits_{(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\mathrm{reg}})}/G_{\mathcal{O}_N})} \fint\limits_{\mathcal{O}'_{N+1}} \int\limits_{\mathfrak{g}_{\mathcal{O}_N\,p}} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\, dX\\
&\quad \cdot \chi_{U_{\mathcal{O}_{N+1}}}(p)\, d\mathcal{O}'_{N+1}(p)\, d(\Omega_{M_{\mathcal{O}_N}(H^{\Omega}_{\mathrm{reg}})}/G_{\mathcal{O}_N})(\mathcal{O}'_{N+1}),
\end{aligned} \tag{7.11}$$

where we used (7.1). Now, recall the observations (5.25) and (5.30) which tell us for each $j\in\{0,\dots,N\}$ how the supports of the amplitudes in the j-th iteration are related to the supports of the amplitudes in the $(j+1)$-th iteration. From (5.25) and (5.30), it follows that one has

$$\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_j}} \operatorname{supp} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N} = \operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_j}} \operatorname{supp} a \qquad \forall\, j\in\{1,\dots,N+1\} \tag{7.12}$$

Taking into account that $\|a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}\|_\infty \le \|a\|_\infty$ holds, we get from (7.12) and (7.11) the estimate

$$\begin{aligned}
&\left|R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N)\right|\\
&\qquad \le \widehat{C}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \|a\|_\infty \operatorname{vol}\left(\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_N}} \operatorname{supp} a\right)\left(1+\nu_N^{\dim\mathfrak{m}_{\mathcal{O}_{N+1}}}\right) \qquad \forall\,\nu_N>0,
\end{aligned} \tag{7.13}$$

where $\widehat{C}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} > 0$ is some constant that is independent of the amplitude a and the indices $l_1,\dots,l_N$. On the other hand, by (5.23), we have the following estimate in the limit $\nu_N\to 0^+$:

$$\begin{aligned}
&R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N)\\
&\quad = \mathrm{O}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\Bigg(\nu_N^{\dim\mathfrak{m}_{\mathcal{O}_{N+1}}+1} \operatorname{vol}\Big(\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_{N+1}}}\big(\operatorname{supp} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}\big)\Big)\\
&\qquad \cdot \sum_{|\alpha|\le \dim\mathfrak{m}_{\mathcal{O}_{N+1}}+3} \sup_{X\in\mathfrak{g}_{\mathcal{O}_{N+1}}} \left\|\partial^\alpha_{B,\xi} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}(\cdot,0,X+\cdot)\right\|_{\mathrm{L}^2(\mathfrak{m}_{\mathcal{O}_{N+1}}\times\mathfrak{m}^*_{\mathcal{O}_{N+1}})}\Bigg),
\end{aligned} \tag{7.14}$$

where the implicit constants in the estimate depend on the tuple $(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})$ but not on any of the amplitudes a, $a_{\mathcal{O}_1}$, $a^{0,l_1}_{\mathcal{O}_1}$, $\dots$, $a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}$. In particular, the estimate is uniform in the variables $l_1,\dots,l_N$. Since we do not know how the occurring $\mathrm{L}^2(\mathfrak{m}_{\mathcal{O}_{N+1}}\times\mathfrak{m}^*_{\mathcal{O}_{N+1}})$-norms are inherited from one iteration step to the next, but we have precise control about this inheritance concerning the supports and the supremum norms of the amplitudes, we now weaken the remainder estimate according to

$$
\begin{aligned}
\text{(7.15)}\quad &\left\|\partial^\alpha_{B,\xi}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}(\cdot,0,X+\cdot)\right\|_{\mathrm{L}^2(\mathfrak{m}_{\mathcal{O}_{N+1}}\times\mathfrak{m}^*_{\mathcal{O}_{N+1}})}\\
&\leq\sqrt{\mathrm{vol}\left(\mathrm{pr}_{\mathfrak{m}^*_{\mathcal{O}_{N+1}}\times\mathfrak{m}_{\mathcal{O}_{N+1}}}\left(\mathrm{supp}\ a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}\right)\right)}\left\|\partial^\alpha_{B,\xi}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}\right\|_\infty\quad\forall\,X\in\mathfrak{g}_{\mathcal{O}_{N+1}}.
\end{aligned}
$$

Since $\mathrm{pr}_{\mathfrak{m}^*_{\mathcal{O}_{N+1}}}\left(\mathrm{supp}\ a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}\right)$ is contained in a compact set determined by cutoff functions which are independent of the amplitudes a, $a_{\mathcal{O}_1}$, $a^{0,l_1}_{\mathcal{O}_1}$, $\dots$, $a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}$, the estimates (7.14), (7.15) and (7.12) imply the following estimate as $\nu_N\to 0^+$:

$$
\begin{aligned}
&R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N)\\
\text{(7.16)}\quad &=\mathrm{O}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\Bigg(\nu_N^{\dim\mathfrak{m}_{\mathcal{O}_{N+1}}+1}\sum_{|\alpha|\leq\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}+3}\left\|\partial^\alpha_{B,\xi}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}\right\|_\infty\mathrm{vol}\left(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_{N+1}}}\mathrm{supp}\ a\right)\\
&\cdot\sqrt{\mathrm{vol}\left(\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}_{N+1}}}\mathrm{supp}\ a\right)}\Bigg),
\end{aligned}
$$

where the estimate is uniform in the amplitudes a, $a_{\mathcal{O}_1}$, $a^{0,l_1}_{\mathcal{O}_1}$, $\dots$, $a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}$ and in particular in the indices $l_1,\dots,l_N$, and the differential operator $\partial^\alpha_{B,\xi}$ is given by

$$
\partial^\alpha_{B,\xi}=\frac{\partial^{\alpha_1}}{\partial B_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}}}{\partial B_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}}}\frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}}}{\partial \xi_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}}}
$$

for an orthonormal basis $\{B_1,\dots,B_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}\}$ of $\mathfrak{m}_{\mathcal{O}_{N+1}}$ with dual basis $\{\xi_1,\dots,\xi_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}\}$. Now, recall the observations (5.31) and (5.32), by which pulling back differential operators on $M_{\mathcal{O}_j}\times\mathfrak{g}_{\mathcal{O}_j}$ along the symplectomorphisms $\varphi_{\mathcal{O}_j}$ in the iteration steps $j\in\{1,\dots,N\}$ leaves the Lie algebra derivatives in the differential operators unaffected. This implies for each multiindex α that there is some differential operator $\mathcal{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ on $M_{\mathcal{O}_0}\times\mathfrak{g}_{\mathcal{O}_0}$ of order $|\alpha|$ which is independent of $l_1,\dots,l_N$ and the amplitude a and contains only derivatives in the $M_{\mathcal{O}_0}$-directions such that

$$
\begin{aligned}
\text{(7.17)}\quad \left\|\partial^\alpha_{B,\xi}a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}\right\|_\infty&\leq\left\|\mathcal{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\partial^\alpha_B\,a|_{U_{\mathcal{O}_1}\times\mathfrak{g}_{\mathcal{O}_0}}\right\|_\infty\\
&=\left\|\mathcal{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\frac{\partial^{\alpha_1}}{\partial B_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}}}{\partial B_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}^{\alpha_{\dim\,\mathfrak{m}_{\mathcal{O}_{N+1}}}}}\,a|_{U_{\mathcal{O}_1}\times\mathfrak{g}_{\mathcal{O}_0}}\right\|_\infty\qquad\forall\,l_1,\dots,l_N\in\{0,1,2,\dots\}.
\end{aligned}
$$

Theorem 6.2 implies that $\dim\mathfrak{m}_{\mathcal{O}_{N+1}}=\dim\mathfrak{g}_{\mathcal{O}_N}-\dim H^\Omega_{\mathrm{reg}}$, so that $\mathcal{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ is a differential operator on M of order $\kappa_{\mathcal{O}_N}+3$, where $\kappa_{\mathcal{O}_N}$ is the dimension of the regular orbits in $\Omega_{M_{\mathcal{O}_N}}$. Concerning the differential operator ∂^α_B, recall from the construction of the model space in Section 4.2 that $\mathfrak{m}_{\mathcal{O}_{N+1}}=\mathfrak{g}^\perp_{\mathcal{O}_{N+1}}\subset\mathfrak{g}_{\mathcal{O}_N}$, where $\mathfrak{g}_{\mathcal{O}_{N+1}}$ is the stabilizer algebra of some chosen point $p_{\mathcal{O}_{N+1}}\in\mathcal{O}_{N+1}$. Since the maps $\varphi_{\mathcal{O}_1},\dots,\varphi_{\mathcal{O}_{N+1}}$ are equivariant, the stabilizer algebra of the point $p_{\mathcal{O}_{N+1}}$ agrees with the stabilizer algebra of the point

$$
p^0_{\mathcal{O}_{N+1}}:=\varphi^{-1}_{\mathcal{O}_1}\circ\cdots\circ\varphi^{-1}_{\mathcal{O}_{N+1}}(p_{\mathcal{O}_{N+1}})\in\mathcal{O}_1\subset\Omega_{M_{\mathcal{O}_0(H^\Omega_{\mathrm{reg}})}}\cap U_{\mathcal{O}_1}\subset M_{\mathcal{O}_0(H^\Omega_{\mathrm{reg}})}.
$$

Note that $g_{\mathcal{O}_{N+1}}\subset\mathfrak{g}_{\mathcal{O}_0}$ might be much smaller than $\mathfrak{g}_{\mathcal{O}_0}$, which means that the orthogonal complement of $\mathfrak{g}_{\mathcal{O}_{N+1}}$ in $\mathfrak{g}_{\mathcal{O}_0}$ might be much larger than the orthogonal complement of $\mathfrak{g}_{\mathcal{O}_{N+1}}$ in $\mathfrak{g}_{\mathcal{O}_N}$. In any case, one has the relation

$$
\text{(7.18)}\qquad \mathfrak{m}_{\mathcal{O}_{N+1}}\subset\mathfrak{g}^\perp_{\mathcal{O}_{N+1}}\subset\mathfrak{g}_{\mathcal{O}_0},
$$

where the symbol $\perp$ denotes the orthogonal complement in $\mathfrak{g}_{\mathcal{O}_0}$ from now on. The isotropy algebra of a different point $p' = g' \cdot p^0_{\mathcal{O}_{N+1}} \in \mathcal{O}_1$ is related to that of $p^0_{\mathcal{O}_{N+1}}$ by

$$(\mathfrak{g}_{\mathcal{O}_0})_{g' \cdot p^0_{\mathcal{O}_{N+1}}} = \mathrm{Ad}\,(g'^{-1})\big((\mathfrak{g}_{\mathcal{O}_0})_{p^0_{\mathcal{O}_{N+1}}}\big) = \mathrm{Ad}\,(g')^{-1}(\mathfrak{g}_{\mathcal{O}_{N+1}}).$$

Since our chosen inner product on $\mathfrak{g}_{\mathcal{O}_0}$ is $\mathrm{Ad}\,(G_{\mathcal{O}_0})$-invariant, one has

$$\mathrm{Ad}\,(g')^{-1}(\mathfrak{g}^{\perp}_{\mathcal{O}_{N+1}}) = \mathrm{Ad}\,(g')^{-1}(\mathfrak{g}_{\mathcal{O}_{N+1}})^{\perp} \subset \mathfrak{g}_{\mathcal{O}_0}.$$

We see that, for any point $p \in \mathcal{O}_1$, we can find an element $g \in G_{\mathcal{O}_0}$ such that

$$(g_{\mathcal{O}_0})^{\perp}_p \supset \mathrm{Ad}\,(g)\big(\mathrm{span}\,\{B_1, \dots, B_{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}}\}\big) = \mathrm{span}\,\{\mathrm{Ad}\,(g)B_1, \dots, \mathrm{Ad}\,(g)B_{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}}\}. \tag{7.19}$$

Let us now define a collection of differential operators on $M_{\mathcal{O}_0(H^{\Omega}_{\mathrm{reg}})} \times \mathfrak{g}_{\mathcal{O}_0}$ by

$$\partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}}(p, X) := \frac{\partial^{\alpha_1}}{\partial Y_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{\dim\,(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_p}}}{\partial Y_{\dim\,(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_p}^{\alpha_{\dim\,(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_p}}}, \qquad (p, X) \in M_{\mathcal{O}_0(H^{\Omega}_{\mathrm{reg}})} \times \mathfrak{g}_{\mathcal{O}_0}, \tag{7.20}$$

where $\{Y_1, \dots, Y_{\dim\,(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_p}\}$ is an orthonormal basis of $(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_p \subset \mathfrak{g}_{\mathcal{O}_0}$, and α is a multiindex of dimension $\dim\,(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_p$. Note that the differential operators $\partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}}$ depend neither on the tuple $(\mathcal{O}_0, \dots, \mathcal{O}_{N+1})$ nor on any of the amplitudes a, $a_{\mathcal{O}_1}$, $a^{0,l_1}_{\mathcal{O}_1}$, $\dots$, $a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}$. Since $G_{\mathcal{O}_0}$ is compact and $U_{\mathcal{O}_1} \subset M_{\mathcal{O}_0(H^{\Omega}_{\mathrm{reg}})}$ is pre-compact, it follows from (7.17), (7.18), and (7.19) that one has for each multiindex α the estimate

$$\Big\| \mathcal{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \partial^{\alpha}_B\, a|_{U_{\mathcal{O}_1} \times \mathfrak{g}_{\mathcal{O}_0}} \Big\|_{\infty} \leq C^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \Big\| \mathcal{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}}\, a|_{M_{\mathcal{O}_0(H^{\Omega}_{\mathrm{reg}})} \times \mathfrak{g}_{\mathcal{O}_0}} \Big\|_{\infty} \quad \forall\, l_1, \dots, l_N \in \{0, 1, 2, \dots\}$$

for some constant $C^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} > 0$. Absorbing the constant according to

$$\mathfrak{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} := C^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \mathcal{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}},$$

and recalling (7.17) and (7.16), we arrive at the remainder estimate in the limit $\nu_N \to 0^+$

$$R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N) = \mathrm{O}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\Bigg(\nu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1} \sum_{|\alpha| \leq \dim\, \mathfrak{m}_{\mathcal{O}_{N+1}}+3} \Big\| \mathfrak{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}}\, a|_{M_{\mathcal{O}_0(H^{\Omega}_{\mathrm{reg}})} \times \mathfrak{g}_{\mathcal{O}_0}} \Big\|_{\infty} \mathrm{vol}\,(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_{N+1}}} \mathrm{supp}\ a) \sqrt{\mathrm{vol}\,(\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}_{N+1}}} \mathrm{supp}\ a)} \Bigg) \tag{7.21}$$

which is uniform in $l_1, \dots, l_N$ and a. Here, the differential operators $\partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}}$ are defined by (7.20), and the differential operators $\mathfrak{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ of order $|\alpha|$ contain only derivatives with respect to the $M_{\mathcal{O}_0}$-directions. The statement (7.21) means that there is a constant $C_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} > 0$ which is independent of $l_1, \dots, l_N$ and the amplitude a, such that one has for all $\nu_N \leq 1$ the inequality

$$\begin{aligned} \big| R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N) \big| &\leq \nu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1} C_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \\ \sum_{|\alpha| \leq \dim\, \mathfrak{m}_{\mathcal{O}_{N+1}}+3} &\Big\| \mathfrak{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}} a|_{M_{\mathcal{O}_0}(H^{\Omega}_{\mathrm{reg}}) \times \mathfrak{g}_{\mathcal{O}_0}} \Big\|_{\infty} \mathrm{vol}\,(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_{N+1}}} \mathrm{supp}\ a) \sqrt{\mathrm{vol}\,(\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}_{N+1}}} \mathrm{supp}\ a)} \\ &=: \nu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1} C^{a}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}. \end{aligned} \tag{7.22}$$

If $N = 0$, then $\nu_N = \mu$ and the result (7.21) already represents the required remainder estimate. Therefore, assume from now on that $N \geq 1$. This can happen only if

$$\dim \mathrm{Reg}\, M_{\mathrm{red}} > 0. \tag{7.23}$$

For technical purposes, we also assume that $\mu \leq 1$. Then, if the numbers $l_1, \dots, l_N$ fulfill

$$\nu_N = \mu \prod_{q=1}^{N} \tau_{l_q}^{-2} \leq 1, \tag{7.24}$$

one has for all $\delta \geq 0$ the estimate

$$\begin{aligned} &\left| \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}} \left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2} \right) \right| \\ &\qquad \leq C^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \mu^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}} + 1 - \delta} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N+1} \dim \mathfrak{m}_{\mathcal{O}_j} - 2 + 2\delta}. \end{aligned} \tag{7.25}$$

We claim that the statements

$$\begin{aligned} &\dim \Omega_{M_{\mathcal{O}_q}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_q} \\ &\qquad = \dim \Omega_{M_{\mathcal{O}_0}(H^\Omega_{\mathrm{reg}})/G_{\mathcal{O}_0}} - \dim \Omega_{M_{\mathcal{O}_0}(G_{\mathcal{O}_q})}/G_{\mathcal{O}_0} =: \Gamma(G_{\mathcal{O}_q}), \qquad \forall\, q \in \{1, \dots, N\} \end{aligned} \tag{7.26}$$

$$\Gamma(G_{\mathcal{O}_q}) \geq 2 \qquad \forall\, q \in \{1, \dots, N\}, \tag{7.27}$$

and

$$\dim W_{\mathcal{O}_q} - 2 \sum_{j=q+1}^{N+1} \dim \mathfrak{m}_{\mathcal{O}_j} \geq \Gamma(G_{\mathcal{O}_q}) \qquad \forall\, q \in \{1, \dots, N\} \tag{7.28}$$

are true. Indeed, by (4.46), (4.48) and (4.29), one has for each $q \in \{1, \dots, N\}$ and each closed subgroup $H \subset G_{\mathcal{O}_q}$ the formula

$$\dim \Omega_{M_{\mathcal{O}_q}(H)}/G_{\mathcal{O}_q} = \dim \Omega_{M_{\mathcal{O}_{q-1}}(H)}/G_{\mathcal{O}_{q-1}} - \dim \Omega_{M_{\mathcal{O}_{q-1}}(G_{\mathcal{O}_q})}/G_{\mathcal{O}_{q-1}}.$$

Applying this formula inductively, once for $H = H^\Omega_{\mathrm{reg}}$ and once for $H = G_q$, proves the claim (7.26). To prove the claim (7.27), recall from (7.23) that we have

$$\dim \operatorname{Reg} M_{\mathrm{red}} \equiv \dim \Omega_{M_{\mathcal{O}_0}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_0} > 0.$$

As the dimension of a symplectic manifold, the left hand side above has to be even. Therefore, (7.23) is equivalent to

$$\dim \Omega_{M_{\mathcal{O}_0}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_0} \geq 2.$$

Now, for $q \leq N$, the space $\Omega_{M_{\mathcal{O}_0}(G_{\mathcal{O}_q})}/G_{\mathcal{O}_0}$ is the stratum in M_{red} corresponding to orbits of a type which is strictly smaller than $(H^\Omega_{\mathrm{reg}}) = (G_{\mathcal{O}_{N+1}})$. Since the stratum corresponding to orbits of type $(H^\Omega_{\mathrm{reg}})$ is dense in M_{red} and has positive dimension, the dimension of any other stratum must be strictly lower than that of $M_{\mathrm{red}(H^\Omega_{\mathrm{reg}})}$, which implies

$$\dim \Omega_{M_{\mathcal{O}_0}(G_{\mathcal{O}_q})}/G_{\mathcal{O}_0} < \dim \Omega_{M_{\mathcal{O}_0}(H^\Omega_{\mathrm{reg}})}/G_{\mathcal{O}_0} \qquad \forall\, q \in \{1, \dots, N\}.$$

Again, because both numbers are even, their difference must be at least 2, proving (7.27). To prove (7.28), we use that $\dim \mathfrak{m}_{\mathcal{O}_j} = \dim \mathfrak{g}_{\mathcal{O}_{j-1}} - \dim \mathfrak{g}_{\mathcal{O}_j}$, which yields

$$\sum_{j=q+1}^{N+1} \dim \mathfrak{m}_{\mathcal{O}_j} = \dim \mathfrak{g}_{\mathcal{O}_q} - \dim \mathfrak{g}_{\mathcal{O}_{N+1}} = \text{dim. of orbits in } \Omega_{M_{\mathcal{O}_q}} \text{ of type } (H^\Omega_{\mathrm{reg}}). \tag{7.29}$$

Corollary 4.11 applied to $(M_{\mathcal{O}_q}, G_{\mathcal{O}_q})$ says that no orbit in $\Omega_{M_{\mathcal{O}_q}}$ of type $(H^\Omega_{\mathrm{reg}})$ is of larger dimension than $\frac{1}{2}(\dim M_{\mathcal{O}_q} - \dim \Omega_{M_{\mathcal{O}_q}}(H^\Omega_{\mathrm{reg}})/G_{\mathcal{O}_q})$. However, one has $\dim W_{\mathcal{O}_q} = \dim M_{\mathcal{O}_q}$ because $M_{\mathcal{O}_q}$ is an open subset of $W_{\mathcal{O}_q}$, so that the claim (7.28) follows from (7.29) and (7.26).

Recalling that $\tau_l = 2^{-l}$, we deduce from (7.25) and (7.28) for arbitrary $\delta \geq 0$ the estimate

$$\left| \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\ldots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}} \left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2} \right) \right| \leq C^a_{\mathcal{O}_0,\ldots,\mathcal{O}_{N+1}} \mu^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\delta}\, 2^{-\sum_{q=1}^{N} l_q(\Gamma(G_{\mathcal{O}_q})-2+\delta)}$$

which holds for all numbers $l_1, \ldots, l_N \in \{0, 1, \ldots\}$ such that (7.24) is fulfilled. To proceed, note that the condition (7.24) is equivalent to

$$2^{2\sum_{q=1}^{N} l_q} \leq \frac{1}{\mu} \iff \sum_{q=1}^{N} l_q \leq \frac{1}{2\log 2} \log\left(\mu^{-1}\right). \tag{7.30}$$

We now want to sum the remainder terms over the index $l_N \in \{0, 1, \ldots\}$ for a fixed set of indices $l_1, \ldots, l_{N-1} \in \{0, 1, 2, \ldots\}$. To this end, let us write (7.30) as

$$l_N \leq \frac{1}{2\log 2} \log\left(\mu^{-1}\right) - \sum_{q=1}^{N-1} l_q. \tag{7.31}$$

Then, from the previous estimates, one immediately deduces for the fixed set of indices $l_1, \ldots, l_{N-1} \in \{0, 1, 2, \ldots\}$ and arbitrary $\varepsilon > 0$ the estimate

$$\left| \sum_{l_N=0}^{\lfloor \frac{1}{2\log 2} \log(\mu^{-1}) \rfloor - \sum_{q=1}^{N-1} l_q} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\ldots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}} \left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2} \right) \right| \leq C^a_{\mathcal{O}_0,\ldots,\mathcal{O}_{N+1}} \mu^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon}\, \Xi(\mu, l_1, \ldots, l_{N-1}) 2^{-\sum_{q=1}^{N-1} l_q(\Gamma(G_{\mathcal{O}_q})-2+\varepsilon)}, \tag{7.32}$$

where

$$\Xi(\mu, l_1, \ldots, l_{N-1}) = \begin{cases} 0, & \lfloor \frac{1}{2\log 2} \log(\mu^{-1}) \rfloor - \sum_{q=1}^{N-1} l_q < 0, \\ \left(1 - 2^{-(\Gamma(G_{\mathcal{O}_N})-2+\varepsilon)}\right)^{-1}, & \text{otherwise.} \end{cases}$$

Here, we used that $\tau_l = 2^{-l}$ and applied the elementary estimate

$$\sum_{n=0}^{K} 2^{-\lambda} \leq \sum_{n=0}^{\infty} 2^{-\lambda} = \left(1 - 2^{-\lambda}\right)^{-1} \qquad \forall\, K \in \{0, 1, 2, \ldots\}, \quad \lambda > 0.$$

Remark 7.2. If $\Gamma(G_{\mathcal{O}_N}) > 2$, then it is easy to see that (7.32) holds also with $\varepsilon = 0$.

Next, we want to estimate the sum over l_N from $\lfloor \frac{1}{2\log 2} \log(\mu^{-1}) \rfloor - \sum_{q=1}^{N-1} l_q + 1$ to $+\infty$, still keeping the indices $l_1, \ldots, l_{N-1} \in \{0, 1, 2, \ldots\}$ fixed. Taking (7.28) and (7.13) into account, and assuming now

$$l_N \geq \lfloor \frac{1}{2\log 2} \log(\mu^{-1}) \rfloor - \sum_{q=1}^{N-1} l_q + 1 \qquad \text{(or equivalently } \nu_N > 1\text{)}, \tag{7.33}$$

we get for arbitrary $\varepsilon > 0$, $\varepsilon \leq 1$ the estimate

$$\begin{aligned}
&\left| \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}\left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2}\right)\right| \\
&\leq C_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \|a\|_\infty \operatorname{vol}\left(\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_N}} \operatorname{supp} a\right)\left(1 + \nu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}}\right) \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} \\
&\leq \underbrace{2C_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \|a\|_\infty \operatorname{vol}\left(\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_N}} \operatorname{supp} a\right)}_{:=\widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}} \nu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} \\
&\leq \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \nu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon} \prod_{q=1}^{N} \tau_{l_q}^{2\dim \mathfrak{m}_{\mathcal{O}_{N+1}} + \Gamma(G_{\mathcal{O}_q})} \\
&= \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \mu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon} \prod_{q=1}^{N} \tau_{l_q}^{\Gamma(G_{\mathcal{O}_q})-2+2\varepsilon}.
\end{aligned}$$

Recalling that $\tau_l = 2^{-l}$, we see from (7.27) that the following infinite sum converges absolutely for arbitrary $\varepsilon > 0$, $\varepsilon \leq 1$ and fulfills the estimate

$$\begin{aligned}
&\sum_{l_N = \lfloor \frac{1}{2\log 2} \log(\mu^{-1})\rfloor - \sum_{q=1}^{N-1} l_q + 1}^{\infty} \left| \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}\left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2}\right)\right| \\
&\leq \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \mu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon} \sum_{l_N = \lfloor \frac{1}{2\log 2} \log(\mu^{-1})\rfloor - \sum_{q=1}^{N-1} l_q + 1}^{\infty} \prod_{q=1}^{N} \tau_{l_q}^{\Gamma(G_{\mathcal{O}_q})-2+2\varepsilon} \\
&= \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \mu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon} \sum_{l_N = \lfloor \frac{1}{2\log 2} \log(\mu^{-1})\rfloor - \sum_{q=1}^{N-1} l_q + 1}^{\infty} 2^{-(\Gamma(G_{\mathcal{O}_N})-2+2\varepsilon)} \prod_{q=1}^{N-1} \tau_{l_q}^{\Gamma(G_{\mathcal{O}_q})-2+2\varepsilon} \\
&\leq \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \mu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon} \sum_{l_N=0}^{\infty} 2^{-(\Gamma(G_{\mathcal{O}_N})-2+2\varepsilon)} \prod_{q=1}^{N-1} \tau_{l_q}^{\Gamma(G_{\mathcal{O}_q})-2+2\varepsilon}
\end{aligned}$$

(7.34)

$$= \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \frac{\mu_N^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\varepsilon}}{1 - 2^{-(\Gamma(G_{\mathcal{O}_N})-2+2\varepsilon)}} \prod_{q=1}^{N-1} \tau_{l_q}^{(\Gamma(G_{\mathcal{O}_q})-2+2\varepsilon)}.$$

If $\Gamma(G_{\mathcal{O}_N}) > 2$, we can even choose $\varepsilon = 0$ in these calculations, compare Remark 7.2.

From (7.34) and (7.32), we conclude inductively that one has for all $\mu, \varepsilon_1, \dots, \varepsilon_N \in (0,1)$ the estimate

$$\begin{aligned}
(7.35)\quad &\left| \sum_{l_1,\dots,l_N=0}^{\infty} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N} \dim \mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}\left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2}\right)\right| \\
&\leq \left(C^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} + \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\right) \mu^{\dim \mathfrak{m}_{\mathcal{O}_{N+1}}+1-\sum_{q=1}^{N}\varepsilon_q} \prod_{q=1}^{N} \left(1 - 2^{-(\Gamma(G_{\mathcal{O}_q})-2+2\varepsilon_q)}\right)^{-1}.
\end{aligned}$$

In particular, the infinite sums over the variables $l_1, \dots, l_N$ are absolutely convergent.

Remark 7.3. From the computations above it is clear that if $\Gamma(G_{\mathcal{O}_q}) > 2$ for some $q \in \{1, \dots, N\}$, then (7.35) remains true even if one puts $\varepsilon_q = 0$.

We are now free to choose for each $q \in \{1,\dots,N\}$ with $\Gamma(G_{\mathcal{O}_q})=2$ a function

$$\mu \mapsto \varepsilon_q(\mu) > 0$$

which has the property that

$$\varepsilon_q(\mu) \to 0^+ \qquad \text{as } \mu \to 0^+$$

and which optimizes the estimate (7.35). This happens if, and only if,

$$\mu^{-\varepsilon_q(\mu)} \sim \left(1-2^{-2\varepsilon_q(\mu)}\right)^{-1} \qquad \text{as } \mu \to 0^+. \tag{7.36}$$

To find a function $\mu \to \varepsilon(\mu) =: \varepsilon_q(\mu)$ fulfilling (7.36), we observe that one has in the limit $\lambda \to 0^+$ the relations

$$\frac{1}{1-2^{-2\lambda}} = \frac{1}{1-e^{-2\log(2)\lambda}} = \frac{1}{2\log(2)\lambda + \mathrm{O}(\lambda^2)} \sim 2\log(2)\lambda \sim \lambda, \tag{7.37}$$

so that, for each $N \geq 1$, the condition (7.36) is equivalent to

$$e^{-\log(\mu)\varepsilon(\mu)} \sim e^{-\log\varepsilon(\mu)} \iff \frac{\log\varepsilon(\mu)}{\varepsilon(\mu)} \sim \log\mu.$$

This yields

$$\varepsilon(\mu) \sim e^{-W(\log\mu^{-1})},$$

where $\lambda \mapsto W(\lambda)$ is the product logarithm or so-called *Lambert W-function.* It is well-known (see [5]) that the product logarithm has the asymptotic property

$$W(\lambda) \sim \log\lambda + \mathrm{O}(\log\log\lambda) \qquad \text{as } \lambda \to +\infty,$$

so that we arrive at the result that

$$\varepsilon_q(\mu) := \varepsilon(\mu) := \frac{1}{\log\left(\mu^{-1}\right)}, \qquad q \in \{1,\dots,N\}$$

is an optimal choice for the functions $\mu \mapsto \varepsilon_q(\mu)$ with respect to the remainder estimate (7.35). By Remark 7.3 and formulas (7.36), (7.37), the result (7.35) now turns into

$$\begin{aligned}\Bigg| \sum_{l_1,\dots,l_N=0}^{\infty} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N}\dim\mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}\Big(\mu\prod_{q=1}^{N}\tau_{l_q}^{-2}\Big)\Bigg| \\ \leq \big(C^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} + \widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\big)\mu^{\dim\mathfrak{m}_{\mathcal{O}_{N+1}}+1}\log\left(\mu^{-1}\right)^{\Lambda(G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N})},\end{aligned} \tag{7.38}$$

where

$$\Lambda(G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N}) \leq N$$

is the number of those groups $G_{\mathcal{O}_q}$ in $\{G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N}\}$ such that $\Gamma(G_{\mathcal{O}_q})=2$. Let us now formulate the result (7.38) in terms of the original amplitude a, the manifold $M_{\mathcal{O}_0} \equiv M$, and the group $G_{\mathcal{O}_0} \equiv G$. Recalling the definitions of $C^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ and $\widehat{C}^a_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$, and using (7.9), we arrive at the following result.

Proposition 7.4. *For each $N \in \{0,\dots,\Lambda(a)\}$ and each maximal orbit tuple $(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})$, one has estimate*

$$\begin{aligned} &R^{\text{Case 1}}_{(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})}(\mu) \\ &:= (2\pi\mu)^{\sum_{j=0}^{N}\dim\mathfrak{m}_{\mathcal{O}_j}} \sum_{l_1,\dots,l_N=0}^{\infty} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N}\dim\mathfrak{m}_{\mathcal{O}_j}} R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}\Big(\mu\prod_{q=1}^{N}\tau_{l_q}^{-2}\Big) \\ &= \mathrm{O}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\Bigg(\mu^{\kappa+1}\log\left(\mu^{-1}\right)^{\Lambda(G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N})}\Big[\|a\|_\infty \operatorname{vol}\left(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_N}}\operatorname{supp} a\right) + \\ &\sum_{|\alpha|\leq\kappa+3}\Big\|\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}\partial^\alpha_{(\mathfrak{g}_{\mathcal{O}_0})^\perp_\bullet} a|_{M_{\mathcal{O}_0}(H^\Omega_{\text{reg}})\times\mathfrak{g}_{\mathcal{O}_0}}\Big\|_\infty \operatorname{vol}\left(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_{N+1}}}\operatorname{supp} a\right)\sqrt{\operatorname{vol}\left(\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}_{N+1}}}\operatorname{supp} a\right)}\Big]\Bigg),\end{aligned}$$

where $\kappa = \dim G_{\mathcal{O}_0} - \dim H^{\Omega}_{\text{reg}}$ is the dimension of the regular orbits in Ω (or, equivalently, the principal orbits in M by assumption), the differential operators $\mathfrak{D}^{\alpha}_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ and $\partial^{\alpha}_{(\mathfrak{g}_{\mathcal{O}_0})^{\perp}_{\bullet}}$ of order $|\alpha|$ were introduced above, and, by (7.26), $\Lambda(G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N})$ is the number of those isotropy types $(G_{\mathcal{O}_q})$ in $\{(G_{\mathcal{O}_1}),\dots,(G_{\mathcal{O}_N})\}$ such that the difference between the dimension of the regular stratum of M_{red} and the stratum of M_{red} corresponding to the orbits of type $(G_{\mathcal{O}_q})$ is equal to 2, which is the minimal difference under the assumption 7.23. In particular, the infinite sums over $l_1,\dots,l_N$ are absolutely convergent and the estimate is uniform in the amplitude a. $\square$

The remainder terms $R^{\text{Case 1}}_{(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})}(\mu)$ represent the contributions of the individual first order remainder terms arising in the applications of the stationary phase theorem in *Case 1* iterations of the desingularization process. Thus, we are left with the task of estimating the remainder terms

$$R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}(\mu) := (2\pi\mu)^{\sum_{j=0}^N \dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{\substack{0\le l_j<L_j,\\ 0\le j\le N}} \prod_{q=0}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^N \dim \mathfrak{m}_{\mathcal{O}_j}} I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}\left(\mu \prod_{q=0}^{N} \tau_{l_q}^{-2}\right)$$

which represent the contributions of the cutoff remainder terms arising in the various iterations of the desingularization process.

7.3. **Remainder estimate II.** In this section, we estimate for an arbitrary orbit tuple $(\mathcal{O}_0,\dots,\mathcal{O}_N)$ the remainder term $R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}(\mu)$. Recall from (5.2) that one has for $\nu_N > 0$ and each collection of indices $l_1,\dots,l_N \in \{0,1,2,\dots\}$ the definition

$$I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\nu_N) = \int_{\mathfrak{g}_{\mathcal{O}_N}} \int_{M_{\mathcal{O}_N}} e^{i\psi_{M_{\mathcal{O}_N}}(p,X)/\nu_N} a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(p,X)\chi_{M_{\mathcal{O}_N}-\Omega_{M_{\mathcal{O}_N}}}(p)\, dp\, dX. \tag{7.39}$$

Using again that $\|a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}\|_\infty \le \|a\|_\infty$ holds, and applying (5.25) and (5.30), we obtain the estimate

$$\left|I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\nu_N)\right| \le C'_{\mathcal{O}_0,\dots,\mathcal{O}_N} \|a\|_\infty \operatorname{vol}(\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_N}} \operatorname{supp} a) \qquad \forall\, l_1,\dots,l_N \in \{0,1,\dots\},\ N \ge 1, \tag{7.40}$$

where $C'_{\mathcal{O}_0,\dots,\mathcal{O}_N} > 0$ is a constant that is independent of the amplitudes a, $a_{\mathcal{O}_1}$, $a^{0,l_1}_{\mathcal{O}_1}$, ..., $a^{0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}$, and in particular of the indices $l_1,\dots,l_N$. In addition, in the case $N = 0$, one has the estimate

$$\left|I^{\square}(\nu_0)\right| \le \|a\|_\infty \operatorname{vol}(\operatorname{supp} a). \tag{7.41}$$

These are the "trivial estimates" which hold merely by definition of the integrals. In contrast, for $N \ge 1$, the non-trivial estimates in the limit $\nu_N \to 0^+$ are given by

$$I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\nu_N) = \mathrm{O}_{\mathcal{O}_0,\dots,\mathcal{O}_N}\left(\nu_N^k \cdot \left\|\Delta_{\mathfrak{g}}^{k/2} a\right\|_\infty \cdot \operatorname{vol}\,(\operatorname{pr}_{\mathfrak{g}_{\mathcal{O}_N}} \operatorname{supp} a)\right) \qquad \forall\, k > 0, \tag{7.42}$$

where we have used (5.5), (5.31), (5.32), (5.25), and (5.30). Similarly, in the case $N = 0$, we have

$$I^{\square}(\nu_0) = \mathrm{O}\left(\nu_0^k \cdot \left\|\Delta_{\mathfrak{g}}^{k/2} a\right\|_\infty \cdot \operatorname{vol}\,(\operatorname{supp} a)\right) \qquad \forall\, k > 0. \tag{7.43}$$

Now, for $N \ge 1$, we can carry out the same arguments as in Subsection 7.2 by the following argument. Although $\mathcal{O}_N$ is not necessarily of type $(H^{\Omega}_{\text{reg}})$, Theorem 6.2 implies that we can complete $(\mathcal{O}_0,\dots,\mathcal{O}_N)$ to an orbit tuple $(\mathcal{O}_0,\dots,\mathcal{O}_N,\mathcal{O}_{N+1})$, where $\mathcal{O}_{N+1}$ is some orbit in $\aleph(M_{\mathcal{O}_N})$ of type $(H^{\Omega}_{\text{reg}})$. Setting

$$I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}(\nu_N) := I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\nu_N)$$

and using 7.42 with $k = \dim \mathfrak{m}_{\mathcal{O}_{N+1}} + 1$, we are in a completely analogous situation as in (7.13) and (7.21), the only essential difference being that one needs to replace the symbol $R^1_{\mathcal{O}_1,0,l_1,\dots,\mathcal{O}_N,0,l_N,\mathcal{O}_{N+1}}(\nu_N)$ by $I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_{N+1}}(\nu_N)$ and the trivial estimate (7.13) by (7.41). Therefore, we directly obtain the equivalent version of Proposition 7.4, which reads

Proposition 7.5. *For each $N \in \{0, \dots, \Lambda(a)\}$ and each orbit tuple $(\mathcal{O}_0, \dots, \mathcal{O}_N)$, one has the estimate*

$$\begin{aligned} &R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}(\mu) \\ &= (2\pi\mu)^{\sum_{j=0}^{N}\dim \mathfrak{m}_{\mathcal{O}_j}} \sum_{l_1,\dots,l_N=0}^{\infty} \prod_{q=1}^{N} \tau_{l_q}^{\dim W_{\mathcal{O}_q} - 2\sum_{j=q+1}^{N}\dim \mathfrak{m}_{\mathcal{O}_j}} I^{\square\, 0,l_1,\dots,0,l_N}_{\mathcal{O}_1,\dots,\mathcal{O}_N}(\nu_N)\left(\mu \prod_{q=1}^{N} \tau_{l_q}^{-2}\right) \\ &= \mathrm{O}_{\mathcal{O}_0,\dots,\mathcal{O}_N}\left(\mu^{\kappa+1}\log\left(\mu^{-1}\right)^{\Lambda(G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N})}\mathrm{vol}\,(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_N}}\,\mathrm{supp}\, a)\left[\|a\|_\infty + \sup_{k\le\kappa}\left\|\Delta_{\mathfrak{g}}^{k/2} a\right\|_\infty\right]\right), \end{aligned}$$

where $\kappa = \dim G_{\mathcal{O}_0} - \dim H^{\Omega}_{\mathrm{reg}}$ is the dimension of the regular orbits in $\Omega_{M_{\mathcal{O}_0}}$ (or, equivalently, the principal orbits in M by assumption), and $\Lambda(G_{\mathcal{O}_1}, \dots, G_{\mathcal{O}_N})$ is the number of those isotropy types $(G_{\mathcal{O}_q})$ in $\{(G_{\mathcal{O}_1}), \dots, (G_{\mathcal{O}_N})\}$ such that the difference between the dimension of the principal stratum of M_{red} and the stratum of M_{red} corresponding to the orbits of type $(G_{\mathcal{O}_q})$ is equal to 2. In particular, the infinite sums over $l_1, \dots, l_N$ are absolutely convergent and the estimate is uniform in the amplitude a. □

Collecting everything, we have shown that

$$I(\mu) \;=\; L_0(2\pi\mu)^{\kappa} \;+\; \sum_{N=0}^{\Lambda(a)} \sum_{\substack{\mathcal{O}_j \in \aleph^-(M_{\mathcal{O}_{j-1}}),\\ 0\le j\le N}} \left[\sum_{\mathcal{O}_{N+1}\in\aleph^+(M_{\mathcal{O}_N})} R^{1a}_{(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})}(\mu) + R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}(\mu)\right], \tag{7.44}$$

where the leading term L_0 is given in Proposition 7.10, and the remainder terms are estimated in Propositions 7.4 and 7.5. Consequently, all that is left to do in order to write down a first order asymptotic formula for $I(\mu)$ is summing up the terms $R^{\text{Case 1}}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}$ and $R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}$. From now on, we will not use the technical, iterative notations $M_{\mathcal{O}_0}$, $\Omega_{M_{\mathcal{O}_0}}$, etc. anymore.

8. Statement of the main result

Let us now return to our departing point, that is, the asymptotic behavior of the integral (2.9) with $\eta = 0$. We can now state the main result of this paper.

Theorem 8.1. *Let M be a symplectic manifold and G a compact Lie group with Lie algebra $\mathfrak{g}$ acting on M in a Hamiltonian way with momentum map $J : M \to \mathfrak{g}^*$. Suppose that the Marsden-Weinstein reduced space $M_{red} = J^{-1}(0)/G$ is connected and that the regular orbit type $(H^{\Omega}_{\mathrm{reg}})$ occurring in $\Omega := J^{-1}(0)$ is given by the principal orbit type of M or fulfills $\dim H^{\Omega}_{\mathrm{reg}} = 0$. Consider the oscillatory integral*

$$I(\mu) = \int_M \int_{\mathfrak{g}} e^{i\psi(p,X)/\mu} a(p,X)\, dX\, dM(p), \qquad \mu > 0, \qquad \psi(p,X) = J(p)(X), \qquad a \in \mathrm{C}^{\infty}_{\mathrm{c}}(M\times\mathfrak{g}),$$

where dM is the canonical symplectic volume form on M, and dX an Euclidean measure given by an $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$. Then, $I(\mu)$ has the asymptotic expansion

$$I(\mu) = (2\pi\mu)^{\kappa} L_0 + \mathrm{O}\big(\mu^{\kappa+1}(\log\mu^{-1})^{\Lambda_a}\, C_a\big), \qquad \mu\to 0^+$$

which is uniform in the amplitude a. Here, κ denotes the dimension of the regular G-orbits in Ω (or, equivalently, the principal orbits in M), and the leading coefficient is given by

$$L_0 \;=\; \frac{\mathrm{vol}\, G}{\mathrm{vol}\, H^{\Omega}_{\mathrm{reg}}} \int_{\mathrm{Reg}\, M_{red}} \fint_{\mathcal{O}} \int_{\mathfrak{g}_p} a(p,A)\, dA\, d\mathcal{O}(p)\, d(\mathrm{Reg}\, M_{red})(\mathcal{O}), \tag{8.1}$$

where $\mathfrak{g}_p \subset \mathfrak{g}$ is the stabilizer algebra of the point p, dA is the Euclidean measure induced on $\mathfrak{g}_p$ by the $\mathrm{Ad}\,(G)$-invariant inner product on $\mathfrak{g}$, $d(\mathrm{Reg}\, M_{red})$ is the canonical symplectic volume form on $\mathrm{Reg}\, M_{red}$, and the orbit measure $d\mathcal{O}$ is arbitrary as long as it is G-invariant and non-zero. The constants Λ_a and C_a occurring in (8.1) are defined as follows.

- $\Lambda_a \in \{0,1,2\dots\}$ *denotes the maximal length of a totally ordered chain* $(H_1) > \dots > (H_N)$ *of orbit types that occur in*
$$J^{-1}(0) \cap \mathrm{pr}_M(\mathrm{supp}\ a)$$
and fulfill
$$\dim \mathrm{Reg}\, M_{red} - \dim M_{red(H_j)} = 2 \qquad \forall\, j \in \{1,\dots,N\},$$
where $M_{red(H_j)}$ *denotes the stratum of* M_{red} *consisting of all orbits of type* (H_j).
- *The constant* $C_a > 0$ *can be written as* $C_a = C_{\mathrm{pr}_M(\mathrm{supp}\ a)}\big(C_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)}\, D_a^{\mathfrak{g}} + C'_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)}\, D_a^M\big)$, *where*

  (1) $C_{\mathrm{pr}_M(\mathrm{supp}\ a)} > 0$ *is a constant that depends only on the projection of the support of the amplitude* a *onto* M *and can be chosen uniformly if* $\mathrm{pr}_M(\mathrm{supp}\ a)$ *varies inside some chosen compact set.*
  (2) *The constant* $C_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)}$ *is given by*
$$C_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)} := \sup_{\mathfrak{L}} \mathrm{vol}\left(\mathrm{pr}_{\mathfrak{L}}(\mathrm{supp}\ a)\right),$$
  where the supremum is taken over all Lie subalgebras $\mathfrak{L}$ *of* $\mathfrak{g}$, *equipped with the measure induced by the* $\mathrm{Ad}\,(G)$*-invariant inner product on* $\mathfrak{g}$.
  (3) *The constant* $D_a^{\mathfrak{g}}$ *is given by*
$$D_a^{\mathfrak{g}} = \sup_{k\leq\kappa} \left\|\Delta_{\mathfrak{g}}^{k/2} a\right\|_{\infty},$$
  where $\Delta_{\mathfrak{g}}$ *is the pullback of the Laplace operator on* $\mathfrak{g}$ *to* $M\times\mathfrak{g}$.
  (4) *Similarly to* $C_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)}$, *the constant* $C'_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)}$ *is given by*
$$C'_{\mathrm{pr}_{\mathfrak{g}}(\mathrm{supp}\ a)} := \sup_{\mathfrak{L}} \max\left(\mathrm{vol}\left(\mathrm{pr}_{\mathfrak{L}}(\mathrm{supp}\ a)\right), \mathrm{vol}\left(\mathrm{pr}_{\mathfrak{L}}(\mathrm{supp}\ a)\right)\sqrt{\mathrm{vol}\left(\mathrm{pr}_{\mathfrak{L}^\perp}(\mathrm{supp}\ a)\right)}\,\right),$$
  where $\mathfrak{L}^\perp$ *is the orthogonal complement of the Lie algebra* $\mathfrak{L}$ *in* $\mathfrak{g}$.
  (5) *The constant* D_a^M *is given by*
$$D_a^M = \sup_{|\alpha|\leq\kappa+3} \left\|\mathcal{D}^\alpha \partial^\alpha_{\mathfrak{g}_\bullet^\perp} a|_{M(H^\Omega_{\mathrm{reg}})\times\mathfrak{g}}\right\|_{\infty},$$
  where α *is a multiindex,* $\mathcal{D}^\alpha$ *is a differential operator of order* $|\alpha|$ *on* $M\times\mathfrak{g}$ *that contains only derivatives with respect to the* M*-directions, and* $\partial^\alpha_{\mathfrak{g}_\bullet^\perp}$ *is the differential operator of order* κ *on* $M(H^\Omega_{\mathrm{reg}})\times\mathfrak{g}$ *defined by*
$$\partial^\alpha_{\mathfrak{g}_\bullet^\perp}(p,X) := \frac{\partial^{\alpha_1}}{\partial Y_{p,1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_\kappa}}{\partial Y_{p,\kappa}^{\alpha_\kappa}}, \qquad (p,X) \in M_{(H^\Omega_{\mathrm{reg}})}\times\mathfrak{g},$$
  where $\{Y_{p,1},\dots,Y_{p,\kappa}\}$ *is an orthonormal basis of* $\mathfrak{g}_p^\perp \subset \mathfrak{g}$, *the orthogonal complement of the stabilizer algebra of the point* $p \in M_{(H^\Omega_{\mathrm{reg}})}$. *The differential operators* $\mathcal{D}^\alpha$ *depend indirectly on* $\mathrm{pr}_M(\mathrm{supp}\ a)$ *but can be chosen independently of the amplitude* a *as long as* $\mathrm{pr}_M(\mathrm{supp}\ a)$ *is contained inside some chosen compact set.*

Remark 8.2. Note that equation (8.1) in particular means that the obtained asymptotic expansion for $I(\mu)$ is independent of the explicit desingularization process we used.

Remark 8.3. Let us consider the special case that $\dim H^\Omega_{\mathrm{reg}} = 0$ and the function $b \in C^\infty_c(M)$ defined by $b(p) := a(p,0)$ is G-invariant, so that it can be regarded as a G-invariant compactly supported differential form of degree 0. Then, we have $\mathrm{vol}\, H^\Omega_{\mathrm{reg}} = |H^\Omega_{\mathrm{reg}}|$, the order of the finite group H^Ω_{reg}. Moreover, since the Lie algebra of H^Ω_{reg} is trivial, the stabilizer algebra of points in $\Omega_{(H^\Omega_{\mathrm{reg}})}$ is $\{0\}$. Therefore, the leading term becomes

$$L_0 \;=\; \frac{\mathrm{vol}\, G}{|H^\Omega_{\mathrm{reg}}|} \int\limits_{\mathrm{Reg}\, M_{\mathrm{red}}} \mathcal{K}(b)\, d(\mathrm{Reg}\, M_{\mathrm{red}}), \tag{8.2}$$

where $\mathcal{K} : \Lambda^0_G(M)_c \to \Lambda^0(\operatorname{Reg} M_{\mathrm{red}})$ is the map on 0-forms inducing the Kirwan map (1.2) on cohomology in degree 0.

Proof. Let $C_{\mathrm{pr}_M(\operatorname{supp} a)}$ be the number of orbits $\mathcal{O}_1 \in \aleph(M)$ such that $U_{\mathcal{O}_1} \cap \mathrm{pr}_M(\operatorname{supp} a) \neq \emptyset$. Then, by (7.44) and (7.10), one has

$$
\begin{aligned}
&\big|I(\mu) - L_0\big| \leq C_{\mathrm{pr}_M(\operatorname{supp} a)} \max_{\mathcal{O}_1\in\aleph(M):U_{\mathcal{O}_1}\cap \operatorname{supp} a\neq\emptyset}\ \max_{N\in\{1,\dots,\Lambda(a)\}}\ \max_{(\mathcal{O}_1,\mathcal{O}_2,\dots,\mathcal{O}_N)} \qquad &(8.3)\\
&\qquad C_{(\mathcal{O}_1,\mathcal{O}_2,\dots,\mathcal{O}_N)}\Big(\big|R^{\text{Case 1}}_{(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})}(\mu)\big| + \big|R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}(\mu)\big|\Big), &(8.4)\\
& &(8.5)
\end{aligned}
$$

where $\Lambda(a)$ is the maximal number of a totally ordered subset of the set of isotropy types occurring in $\mathrm{pr}_M(\operatorname{supp} a)$, and $C_{(\mathcal{O}_1,\mathcal{O}_2,\dots,\mathcal{O}_N)} > 0$ are constants that are independent of the amplitude a, since the supports of the new amplitudes produced in the iterations of the desingularization process are contained in the supports of cutoff functions $\chi_\bullet$ which are independent of the amplitude a. In particular, for each $\mathcal{O}_1 \in \aleph(M)$ with $U_{\mathcal{O}_1} \cap \operatorname{supp} a \neq \emptyset$, we need to take the maximum over only a finite set of orbit tuples $(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_N)$, and this finite set depends solely on $\mathcal{O}_1$ and the cutoff functions $\chi_\bullet$ and symplectomorphisms $\varphi_\bullet$ in the desingularization process, but not on the amplitude a. Now, Propositions 7.4 and 7.5 together state that one has for small μ and each of the finitely many orbit tuples $(\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_{N+1})$ occurring in (8.3) the estimate

$$
\begin{aligned}
&\big|R^{\text{Case 1}}_{(\mathcal{O}_0,\dots,\mathcal{O}_{N+1})}(\mu)\big| + \big|R^{\square}_{(\mathcal{O}_0,\dots,\mathcal{O}_N)}(\mu)\big| \leq C'_{(\mathcal{O}_1,\mathcal{O}_2,\dots,\mathcal{O}_{N+1})}\, \mu^{\kappa+1} \log\big(\mu^{-1}\big)^{\Lambda(G_{\mathcal{O}_1},\dots,G_{\mathcal{O}_N})}\\
&\Bigg(\operatorname{vol}(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_N}} \operatorname{supp} a) \sup_{k\leq\kappa}\Big\|\Delta_{\mathfrak{g}}^{k/2} a\Big\|_\infty\\
&+ \sum_{|\alpha|\leq\kappa+3}\Big\|\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \partial^\alpha_{(\mathfrak{g}_{\mathcal{O}_0})^\perp_\bullet} a|_{M_{\mathcal{O}_0(H^\Omega_{\mathrm{reg}})}\times\mathfrak{g}_{\mathcal{O}_0}}\Big\|_\infty \operatorname{vol}\big(\mathrm{pr}_{\mathfrak{g}_{\mathcal{O}_{N+1}}} \operatorname{supp} a\big)\sqrt{\operatorname{vol}\big(\mathrm{pr}_{\mathfrak{m}_{\mathcal{O}_{N+1}}} \operatorname{supp} a\big)}\Bigg),
\end{aligned}
$$

where $C'_{(\mathcal{O}_1,\mathcal{O}_2,\dots,\mathcal{O}_{N+1})} > 0$ is a constant that is independent of the amplitude a, the differential operators $\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$, $\Delta_{\mathfrak{g}}^{k/2}$ and $\partial^\alpha_{(\mathfrak{g}_{\mathcal{O}_0})^\perp_\bullet}$ are independent of the amplitude a, and the numbers $\Lambda(G_{\mathcal{O}_1}, \dots, G_{\mathcal{O}_N})$ are all less or equal to the number Λ_a as defined above. For each multiindex α with $|\alpha| \leq \kappa + 3$, we can now choose a differential operator $\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ which maximizes the number

$$
\Big\|\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}} \partial^\alpha_{(\mathfrak{g}_{\mathcal{O}_0})^\perp_\bullet} a|_{M_{\mathcal{O}_0(H^\Omega_{\mathrm{reg}})}\times\mathfrak{g}_{\mathcal{O}_0}}\Big\|_\infty \tag{8.6}
$$

among all the finitely many differential operators $\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ occurring in the estimate above, and we denote this chosen differential operator by $\mathcal{D}^\alpha_M$. When the projection of the support of a onto M changes, the number of orbit tuples $(\mathcal{O}_0, \dots, \mathcal{O}_{N+1})$ might increase, and in turn the maximal number (8.6) might increase. Hence, although neither of the differential operators $\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ depends on the amplitude a, the maximizer $\mathcal{D}^\alpha_M$ of the number (8.6) depends indirectly on $\mathrm{pr}_M(\operatorname{supp} a)$. However, as long as $\mathrm{pr}_M(\operatorname{supp} a)$ lies inside some fixed compact set $K \subset M$, we can choose a finite number N_K of orbits $\mathcal{O}$ such that the associated open sets $U_{\mathcal{O}}$ cover K, and then one has $N_{\mathrm{pr}_M(\operatorname{supp} a)} \leq N_K$ and we can choose $\mathcal{D}^\alpha_M$ to be the differential operator $\mathfrak{D}^\alpha_{\mathcal{O}_0,\dots,\mathcal{O}_{N+1}}$ that maximizes (8.6) over all orbit tuples $(\mathcal{O}_1, \dots, \mathcal{O}_{N+1})$ with $U_{\mathcal{O}_1} \cap K \neq \emptyset$, a choice that is independent of a. $\square$

Remark 8.4. We would like to close this section by pointing out that the leading term L_0 could alternatively be computed in the following way. By the desingularization process one has

$$
I(\mu) = (2\pi\mu)^d L_0 + \mathrm{O}\big(\mu^{d+1}(\log\mu^{-1})^{\Lambda_a}\, C_a\big), \qquad \mu \to 0^+,
$$

see (8.1), where L_0 is given by the summands that make up the leading term in (6.7), see the beginning of Section 7.1. On the other hand, Proposition 4.7 gives an explicit expression for L_0 if the amplitude

$a(p,X)$ does not intersect the singular strata of the critical set. Let us therefore define $\mathrm{Sing}\,\Omega := \Omega - \mathrm{Reg}\,\Omega$ of Ω. Let K be a compact subset in M, $\delta > 0$, and consider the set

$$(\mathrm{Sing}\,\Omega \cap K)_\delta = \{p \in M : d(p,p') < \delta \text{ for some } p' \in \mathrm{Sing}\,\Omega \cap K\},$$

where d denotes the distance on M corresponding to the Riemannian metric g. Further, let $u_\delta \in \mathrm{C}^\infty_c((\mathrm{Sing}\,\Omega \cap K)_{3\delta})$ be a test function satisfying $u_\delta = 1$ on $(\mathrm{Sing}\,\Omega \cap K)_\delta$. Now, assume that K is such that $\mathrm{supp}_{\,p}\, a \subset K$. We then assert that the limit

$$\lim_{\delta \to 0} \int_{\mathrm{Reg}\,\mathcal{C}} \frac{[a(1-u_\delta)](p,X)}{|\det \psi''(p,X)_{|N_{(p,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} d(\mathrm{Reg}\,\mathcal{C})(p,X) \tag{8.7}$$

exists and is equal to L_0, where $d(\mathrm{Reg}\,\mathcal{C})$ is the measure on $\mathrm{Reg}\,\mathcal{C}$ induced by $dp\,dX$. Indeed, define

$$I_\delta(\mu) := \int_M \int_{\mathfrak{g}} e^{i\psi(p,X)/\mu}[a(1-u_\delta)](p,X)\,dX\,dM(p).$$

Since by Lemma 4.2 $(p,X) \in \mathrm{Sing}\,\mathcal{C}$ implies $p \in \mathrm{Sing}\,\Omega$, an application of Proposition 4.7 for fixed $\delta > 0$ gives

$$|I_\delta(\mu) - (2\pi\mu)^d L_0(\delta)| \leq C_\delta \mu^{d+1}, \tag{8.8}$$

since $d = d - \dim H^\Omega_{\mathrm{reg}}$, where $C_\delta > 0$ is a constant depending only on δ, and

$$L_0(\delta) = \int_{\mathrm{Reg}\,\mathcal{C}} \frac{[a(1-u_\delta)](p,X)}{|\det \psi''(p,X)_{|N_{(p,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} d(\mathrm{Reg}\,\mathcal{C})(p,X).$$

On the other hand, applying the results from the desingularization process to $I_\delta(\mu)$ instead of $I(\mu)$, we obtain again an asymptotic expansion of the form (8.8) for $I_\delta(\mu)$, where now the first coefficient is given by the summands that make up the leading term in (6.7) with the amplitude a replaced by $a(1-u_\delta)$. Since the first term in the asymptotic expansion (8.8) is uniquely determined, the two expressions for $L_0(\delta)$ must be identical. The existence of the limit (8.7) now follows by the Lebesgue theorem on bounded convergence, the corresponding limit being given by L_0. Next, let $a^+ \in \mathrm{C}^\infty_c(M \times \mathfrak{g}, \mathbb{R}^+)$. Since one can assume that $|u_\delta| \leq 1$, the lemma of Fatou implies that

$$\int_{\mathrm{Reg}\,\mathcal{C}} \lim_{\delta \to 0} \frac{[a^+(1-u_\delta)](p,X)}{|\det \psi''(p,X)_{|N_{(p,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} d(\mathrm{Reg}\,\mathcal{C})(p,X)$$

is majorized by the limit (8.7), with a replaced by a^+, and we obtain

$$\int_{\mathrm{Reg}\,\mathcal{C}} \frac{a^+(p,X)}{|\det \psi''(p,X)_{|N_{(p,X)}\mathrm{Reg}\,\mathcal{C}}|^{1/2}} |d(\mathrm{Reg}\,\mathcal{C})(p,X)| < \infty.$$

Choosing a^+ to be equal 1 on a neighborhood of the support of a, and applying the theorem of Lebesgue on bounded convergence to the limit (8.7), we obtain (8.1) by taking into account once more Proposition 4.7. From this it also follows that the leading coefficient can also be expressed as

$$L_0 \;=\; \frac{\mathrm{vol}\,G}{\mathrm{vol}\,H^\Omega_{\mathrm{reg}}} \int_{\mathrm{Reg}\,\Omega} \left[\int_{\mathfrak{g}_p} a(p,X)\,dX \right] \frac{d(\mathrm{Reg}\,\Omega)(p)}{\mathrm{vol}\,\mathcal{O}_p},$$

where $d(\mathrm{Reg}\,\Omega)$ denotes the Riemannian volume measure induced on $\mathrm{Reg}\,\Omega$ by some Riemannian metric on M, and $\mathrm{vol}\,\mathcal{O}_p$ the corresponding Riemannian volume of the orbit through p.

9. Residue formulae

We are now in position to derive residue formulae for general symplectic manifolds. Indeed, as an application of Theorem 8.1, we are able to compute the limit (2.2) in case that the dimension of the regular G-orbits in Ω equals $d = \dim \mathfrak{g}$ or, equivalently, that $\dim H^\Omega_{\mathrm{reg}} = 0$. It corresponds to the leading term in the expansion.

Corollary 9.1. *Assume that* $\dim H^\Omega_{\mathrm{reg}} = 0$. *Let* $\alpha \in \Lambda^*_c(M)$ *be a* G*-invariant differential form of degree* $2n$ *and* $\phi \in \mathcal{S}(\mathfrak{g}^*)$ *have total integral* 1 *and compactly supported Fourier transform* $\hat\phi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathfrak{g})$. *Then, one has with* $\alpha = a\, dM$

$$\lim_{\varepsilon\to 0} \langle \mathcal{F}_\mathfrak{g} L_\alpha, \phi_\varepsilon\rangle = \lim_{\varepsilon\to 0} \int_\mathfrak{g}\int_M e^{iJ_X/\varepsilon}\alpha\, \hat\phi(X)\frac{dX}{\varepsilon^d}$$
$$= \frac{(2\pi)^d \operatorname{vol} G}{|H^\Omega_{\mathrm{reg}}|}\int\limits_{\operatorname{Reg} M_{\mathrm{red}}} \mathcal{K}(a)\ d(\operatorname{Reg} M_{\mathrm{red}}) = \frac{(2\pi)^d \operatorname{vol} G}{|H^\Omega_{\mathrm{reg}}|}\int\limits_{\operatorname{Reg}\Omega} \frac{r(\alpha)}{\operatorname{vol}\mathcal{O}},$$

where $\mathcal{K} : \Lambda^0_G(M)_c \to \Lambda^0(\operatorname{Reg} M_{\mathrm{red}})$ *is the map on* 0*-forms inducing the Kirwan map (1.2) on cohomology in degree* 0, *and* $r : \Lambda^*_c(M) \to \Lambda^{*-d}(\operatorname{Reg}\Omega)$ *is defined locally by (3.3).*

Proof. The function a is necessarily G-invariant, therefore by (2.2) and Theorem 8.1 one deduces

$$L_0 := \lim_{\varepsilon\to 0}\langle \mathcal{F}_\mathfrak{g} L_\alpha, \phi_\varepsilon\rangle = \frac{(2\pi)^d \operatorname{vol} G}{\left|H^\Omega_{\mathrm{reg}}\right|}\int\limits_{\operatorname{Reg} M_{\mathrm{red}}} \fint_{\mathcal{O}} \hat\phi(0) a(p)\, d\mathcal{O}(p)\, d(\operatorname{Reg} M_{\mathrm{red}})(\mathcal{O}).$$

Since $\hat\phi(0) = 1$, the first claimed equality follows by taking into account Remark 8.3. To see the second, assume that α is supported in a neighborhood of Ω. Let $K \subset M$ be a compact subset such that $\operatorname{supp}\alpha \subset K$, and $u_\delta \in \mathrm{C}^\infty_{\mathrm{c}}((\operatorname{Sing}\Omega\cap K)_{3\delta})$ a family of cut-off functions as in Remark 8.4. Denote the normal bundle to $\operatorname{Reg}\mathcal{C}$ by $\nu : N\operatorname{Reg}\mathcal{C} \to \mathcal{C}$, and identify a tubular neighborhood of $\operatorname{Reg}\mathcal{C}$ with a neighborhood of the zero section in $N\operatorname{Reg}\mathcal{C}$. A direct application of the stationary phase theorem for vector bundles then yields with (3.2) and the identification $\operatorname{Reg}\mathcal{C} = \operatorname{Reg}\Omega\times\{0\}$

$$L_0(\delta) := \lim_{\varepsilon\to 0}\int_\mathfrak{g}\int_M e^{iJ_X/\varepsilon}(1-u_\delta)\,\alpha\,\hat\phi(X)\frac{dX}{\varepsilon^d} = \frac{(2\pi)^d \operatorname{vol} G}{|H^\Omega_{\mathrm{reg}}|}\int_{\operatorname{Reg}\Omega}\frac{r((1-u_\delta)\alpha)}{\operatorname{vol}\mathcal{O}},$$

where only the leading term is relevant. Repeating the arguments in the proof of Theorem 8.1 then shows that

$$L_0 = \lim_{\delta\to 0} L_0(\delta) = \frac{(2\pi)^d \operatorname{vol} G}{|H^\Omega_{\mathrm{reg}}|}\int_{\operatorname{Reg}\Omega}\frac{r(\alpha)}{\operatorname{vol}\mathcal{O}},$$

and we obtain the second equality. ☐

After these preparations, we finally arrive at

Theorem 9.2. *Let* M *be a symplectic manifold carrying a Hamiltonian action of a compact, connected Lie group* G *of dimension* d *with momentum map* $J : M \to \mathfrak{g}^*$ *and maximal torus* T. *Assume that the symplectic reduction* M_{red} *is connected, and let* $\operatorname{Reg} M_{\mathrm{red}} := (\Omega\cap M_{(H^\Omega_{\mathrm{reg}})})/G$ *be its regular stratum, where* $\Omega := J^{-1}(0)$. *Suppose that* H^Ω_{reg} *is a finite group. Let* $[\varrho] \in H^*_G(M)_c$ *be of the form* $\varrho(X) = \alpha + D\nu(X)$, *where* α *is a basic differential form of compact support and* ν *an equivariant differential form of compact support. Then the sum*

$$\operatorname{Res}\Big(\Phi^2\sum_{F\in\mathcal{F}} u_F\Big) := \sum_{F\in\mathcal{F}}\operatorname{Res}^{\Lambda,\eta}(u_F\Phi^2), \qquad \eta\in\Lambda\subset\mathfrak{t}^*,$$

of residues, as defined in Definition 2.1, is independent of η *and* Λ, *and one has*

$$(-2\pi i)^d \int\limits_{\operatorname{Reg} M_{\mathrm{red}}} \mathcal{K}(\alpha) e^{-i\omega_{red}} = \frac{|H^\Omega_{\mathrm{reg}}|}{|W|\operatorname{vol} T}\operatorname{Res}\Big(\Phi^2\sum_{F\in\mathcal{F}} u_F\Big),$$

where $\mathcal{K}$ *is the Kirwan map (1.2),* W *is the Weyl group of* $\Delta(\mathfrak{g}^\mathbb{C},\mathfrak{t}^\mathbb{C})$, Φ *denotes the product of the positive roots, and the rational functions* u_F *on the Lie algebra* $\mathfrak{t}$ *of* T *are defined by (2.5).*

Proof. By definition, α is a compactly supported, G-invariant differential form on M that is closed and satisfies $\iota_{\widetilde{X}}\alpha = 0$ for all $X\in\mathfrak{g}$. In particular, viewed as an equivariant differential form, α is a polynomial of degree 0 on $\mathfrak{g}^*$. Let $2n = \dim M$ and set

$$\operatorname{codeg}\alpha := \dim M - \deg\alpha = 2n - \deg\alpha,$$

where $\deg\alpha$ is the degree of α. Now, for any $\eta\in\mathfrak{g}^*$ and $\varepsilon>0$ we have

$$\int_{\mathfrak{g}}\left[\int_M e^{i(J-\eta)(X)}e^{-i\omega}D\nu(X)\right]\hat{\phi}_\varepsilon(X)dX=0,$$

see [21, Lemma 1]. With Corollary 9.1 it therefore follows that

$$\lim_{\varepsilon\to 0}\int_{\mathfrak{g}}\int_M e^{iJ(X)/\varepsilon}e^{-i\omega}\varrho(X/\varepsilon)\,\hat{\phi}(X)\frac{dX}{\varepsilon^d}=\frac{(2\pi)^d\operatorname{vol}G}{|H^\Omega_{\mathrm{reg}}|}\int\limits_{\operatorname{Reg}M_{\mathrm{red}}}\mathcal{K}(a)\;d(\operatorname{Reg}M_{\mathrm{red}}),$$

where the G-invariant function $a\in \mathrm{C}^\infty_{\mathrm{c}}(M)$ vanishes if $\deg\alpha\notin 2\mathbb{N}\cup\{0\}$, and is otherwise defined by

$$\alpha\wedge\frac{(-i\omega)^{\operatorname{codeg}\alpha/2}}{(\operatorname{codeg}\alpha/2)!}=a\;dM\equiv a\,\frac{\omega^n}{n!}. \tag{9.1}$$

In view of Proposition 2.3 and the arguments given in the proof of (4.7), it remains to prove

$$\int\limits_{\operatorname{Reg}M_{\mathrm{red}}}\mathcal{K}(a)\;d(\operatorname{Reg}M_{\mathrm{red}})=(-i)^d\int\limits_{\operatorname{Reg}M_{\mathrm{red}}}\mathcal{K}(\alpha)e^{-i\omega_{\mathrm{red}}}. \tag{9.2}$$

Only the top degree component is being integrated on the right hand side of (9.2), so that

$$\int\limits_{\operatorname{Reg}M_{\mathrm{red}}}\mathcal{K}(\alpha)e^{-i\omega_{\mathrm{red}}}=\begin{cases}\displaystyle\int\limits_{\operatorname{Reg}M_{\mathrm{red}}}\mathcal{K}(\alpha)\wedge\frac{(-i\omega_{\mathrm{red}})^{\operatorname{codeg}\mathcal{K}(\alpha)/2}}{(\operatorname{codeg}\mathcal{K}(\alpha)/2)!}, & \deg\mathcal{K}(\alpha)\in 2\mathbb{N}\cup\{0\},\\ 0, & \deg\mathcal{K}(\alpha)\notin 2\mathbb{N}\cup\{0\},\end{cases} \tag{9.3}$$

where

$$\begin{aligned}\operatorname{codeg}\mathcal{K}(\alpha):&=\dim\operatorname{Reg}M_{\mathrm{red}}-\deg\mathcal{K}(\alpha)=\dim\operatorname{Reg}M_{\mathrm{red}}-\deg\alpha=\dim M-2d-\deg\alpha\\ &=\operatorname{codeg}\alpha-2d.\end{aligned} \tag{9.4}$$

From (9.4), we see that if $\deg\alpha\notin 2\mathbb{N}\cup\{0\}$, both sides of the equation (9.2) are equal to 0, in particular the equation holds. Thus, suppose in the following that $\deg\alpha=:2p\in 2\mathbb{N}\cup\{0\}$. Then, in order to show (9.2), it suffices to prove the statement

$$(-i)^d\mathcal{K}(\alpha)\wedge\frac{(-i\omega_{\mathrm{red}})^{n-d-p}}{(n-d-p)!}=\mathcal{K}(a)\;d(\operatorname{Reg}M_{\mathrm{red}})\equiv\mathcal{K}(a)\;\frac{\omega_{\mathrm{red}}^{n-d}}{(n-d)!}. \tag{9.5}$$

Now, since ω_{red} is characterized by the relation $\iota^*_{\operatorname{Reg}\Omega}\omega=\pi^*\omega_{\mathrm{red}}$, where $\iota_{\operatorname{Reg}\Omega}:\Omega_{(H^\Omega_{\mathrm{reg}})}\to M$ is the inclusion and $\pi:\Omega_{(H^\Omega_{\mathrm{reg}})}\to\operatorname{Reg}M_{\mathrm{red}}$ is the canonical projection (cf. [20, Theorem 8.1.1]), we have $(\iota_{\operatorname{Reg}\Omega}\circ j)^*(\omega|_{\pi^{-1}(U)})=\omega_{\mathrm{red}}|_U$ for any local section $j:U\to\pi^{-1}(U)\subset\Omega_{(H^\Omega_{\mathrm{reg}})}$ of π, and since $\mathcal{K}$ agrees locally with $(\iota_{\operatorname{Reg}\Omega}\circ j)^*$ on basic forms and the wedge product commutes with pullbacks, (9.5) is equivalent to the statement that one has for every small open set $U\subset\operatorname{Reg}M_{\mathrm{red}}$ the equality

$$(\iota_{\operatorname{Reg}\Omega}\circ j)^*\left((-i)^d\alpha\wedge\frac{(-i\omega_{\mathrm{red}})^{n-d-p}}{(n-d-p)!}\bigg|_{(\pi\circ\iota_{\operatorname{Reg}\Omega})^{-1}(U)}\right)=(\iota_{\operatorname{Reg}\Omega}\circ j)^*\left(a\,\frac{\omega_{\mathrm{red}}^{n-d}}{(n-d)!}\bigg|_{(\pi\circ\iota_{\operatorname{Reg}\Omega})^{-1}(U)}\right). \tag{9.6}$$

Thus, in order to prove (9.2), it suffices to prove

$$(-i)^d\,\iota^*_{\operatorname{Reg}\Omega}\alpha\wedge\frac{(-i\,\iota^*_{\operatorname{Reg}\Omega}\omega)^{n-d-p}}{(n-d-p)!}=a|_{\Omega_{(H^\Omega_{\mathrm{reg}})}}\,\frac{\iota^*_{\operatorname{Reg}\Omega}\omega^{n-d}}{(n-d)!}. \tag{9.7}$$

We will prove (9.7) using the local normal form of the momentum map. To this end, let $p\in\Omega_{(H^\Omega_{\mathrm{reg}})}$ be a point with isotropy group H^Ω_{reg}, and let $\mathcal{O}\subset\Omega_{(H^\Omega_{\mathrm{reg}})}$, $\mathcal{O}\cong G/H^\Omega_{\mathrm{reg}}$, be its G-orbit. By Theorem 4.10, there is a G-invariant open neighbourhood $U_{\mathcal{O}}$ of $\mathcal{O}$ in M and an open neighbourhood $\mathcal{U}_{\mathcal{O}}$ of the zero section in $\mathcal{Y}_{\mathcal{O}}=(G\times\mathfrak{m}^*\times V)/H^\Omega_{\mathrm{reg}}$ together with a G-equivariant symplectomorphism $\varphi_{\mathcal{O}}:U_{\mathcal{O}}\xrightarrow{\cong}\mathcal{U}_{\mathcal{O}}$. Here, V is the symplectic slice at p, a symplectic vector space, and $\mathfrak{m}\oplus\mathfrak{h}=\mathfrak{g}$, where $\mathfrak{h}$ is the Lie algebra of H^Ω_{reg}. The situation in the local normal form theorem is much simpler when considering a regular orbit than in the general case. Indeed, since H^Ω_{reg} is 0-dimensional, we have $\mathfrak{h}=\{0\}$, so $\mathfrak{m}=\mathfrak{g}$. In

particular, the momentum map associated to the H^Ω_{reg}-action on V is constantly 0, and the H^Ω_{reg}-equivariant diffeomorphism (4.22) is the identity (up to the trivial identification $G\times\mathfrak{g}^*\times\{0\}\times V = G\times\mathfrak{g}^*\times V$). Moreover, by (4.27) and (4.28), the stratum of type $(H^\Omega_{\mathrm{reg}})$ of the zero level of the momentum map $\mathcal{J}_{\mathcal{O}}$ on $\mathcal{Y}_{\mathcal{O}}$ corresponds to the linear symplectic subspace $V_{H^\Omega_{\mathrm{reg}}}$ of V given by those vectors which are fixed by the H^Ω_{reg}-action. However, by counting dimensions one easily sees that $V_{H^\Omega_{\mathrm{reg}}}$ needs to have the same dimension as V, and consequently agrees with V, i.e. H^Ω_{reg} acts trivially on V. Note that since $\dim H^\Omega_{\mathrm{reg}}=0$, we can work on the space $G\times\mathfrak{g}^*\times V$ when considering differential forms, since the tangent spaces of $G\times\mathfrak{g}^*\times V$ and $\mathcal{Y}_{\mathcal{O}}=(G\times\mathfrak{g}^*\times V)/H^\Omega_{\mathrm{reg}}$ agree. These observations imply that the symplectic form on the model space $\mathcal{Y}_{\mathcal{O}}$, or equivalently on $G\times\mathfrak{g}^*\times V$, is simply given by

$$\omega_{\mathcal{Y}_{\mathcal{O}}} = \mathrm{pr}^*_{T^*G}\,\omega_{T^*G} + \mathrm{pr}^*_V\,\omega_V, \tag{9.8}$$

where $\mathrm{pr}_\bullet$ denotes the projection onto $\bullet$, and since $V_{H^\Omega_{\mathrm{reg}}}=V$, (4.27) yields

$$\iota^*_{\mathrm{Reg}\,\mathcal{J}_{\mathcal{O}}^{-1}(0)}\omega_{\mathcal{Y}_{\mathcal{O}}} = \mathrm{pr}^*_V\,\omega_V, \tag{9.9}$$

where $\iota_{\mathrm{Reg}\,\mathcal{J}_{\mathcal{O}}^{-1}(0)}:\mathcal{J}_{\mathcal{O}}^{-1}(0)_{(H^\Omega_{\mathrm{reg}})}\to\mathcal{Y}_{\mathcal{O}}$ is the inclusion, and pr_V is now the projection onto V in the space $\mathcal{J}_{\mathcal{O}}^{-1}(0)_{(H^\Omega_{\mathrm{reg}})}\cong G/H^\Omega_{\mathrm{reg}}\times V$. From (9.8), we get the relation

$$\frac{\big(\mathrm{pr}^*_V\,\omega_V\big)^{n-d}}{(n-d)!}\wedge\frac{\big(\mathrm{pr}^*_{T^*G}\,\omega_{T^*G}\big)^{d}}{d!} = \frac{\omega^n_{\mathcal{Y}_{\mathcal{O}}}}{n!}. \tag{9.10}$$

By Theorem 4.10 and (9.10), the defining property (9.1) of the function a can be written over $U_{\mathcal{O}}$ as

$$(\varphi_{\mathcal{O}}^{-1})^*\alpha\wedge\frac{(-i\omega_{\mathcal{Y}_{\mathcal{O}}})^{n-p}}{(n-p)!} = a\circ\varphi_{\mathcal{O}}^{-1}\,\frac{\omega^n_{\mathcal{Y}_{\mathcal{O}}}}{n!} = a\circ\varphi_{\mathcal{O}}^{-1}\,\frac{\big(\mathrm{pr}^*_V\,\omega_V\big)^{n-d}}{(n-d)!}\wedge\frac{\big(\mathrm{pr}^*_{T^*G}\,\omega_{T^*G}\big)^{d}}{d!}. \tag{9.11}$$

Being a basic differential form, α satisfies $\iota_{\widetilde{X}}\alpha=0$ for all $X\in\mathfrak{g}$. Due to the construction of the G-action on $\mathcal{Y}_{\mathcal{O}}$ and Theorem 4.10, this implies that the fiber-wise kernel of the form $(\varphi_{\mathcal{O}}^{-1})^*\alpha$ includes the whole tangent space of the Lie group G. In other words, the support of $(\varphi_{\mathcal{O}}^{-1})^*\alpha$ is fiberwise disjoint from that of $\mathrm{pr}^*_{T^*G}\,\omega_{T^*G}$. Thus, the ω_{T^*G}-powers in (9.11) can arise on the left hand side solely from the powers of $\omega_{\mathcal{Y}_{\mathcal{O}}}$. As d is the maximal power of ω_{T^*G}, the relation (9.11) implies

$$(\varphi_{\mathcal{O}}^{-1})^*\alpha\wedge\frac{(-i\omega_{\mathcal{Y}_{\mathcal{O}}})^{n-p}}{(n-p)!} = (-i)^d\,(\varphi_{\mathcal{O}}^{-1})^*\alpha\wedge\frac{\big(-i\,\mathrm{pr}^*_V\,\omega_V\big)^{n-d-p}}{(n-d-p)!}\wedge\frac{\big(\mathrm{pr}^*_{T^*G}\,\omega_{T^*G}\big)^{d}}{d!}. \tag{9.12}$$

Since the supports of $(\varphi_{\mathcal{O}}^{-1})^*\alpha$ and $\mathrm{pr}^*_V\,\omega_V$ are fiberwise disjoint from that of $\mathrm{pr}^*_{T^*G}\,\omega_{T^*G}$, wedging them with $\mathrm{pr}^*_{T^*G}\,\omega_{T^*G}$ is an injective operation and we can deduce from (9.11) and (9.12) the result

$$(-i)^d\,(\varphi_{\mathcal{O}}^{-1})^*\alpha\wedge\frac{\big(-i\,\mathrm{pr}^*_V\,\omega_V\big)^{n-d-p}}{(n-d-p)!} = a\circ\varphi_{\mathcal{O}}^{-1}\,\frac{\big(\mathrm{pr}^*_V\,\omega_V\big)^{n-d}}{(n-d)!}. \tag{9.13}$$

Taking the pullback along $\iota_{\mathrm{Reg}\,\Omega}$ and using (9.9), we arrive at the result

$$(-i)^d\,(\varphi_{\mathcal{O}}^{-1}\circ\iota_{\mathrm{Reg}\,\Omega})^*\alpha\wedge\frac{\big(-i\,\iota^*_{\mathrm{Reg}\,\mathcal{J}_{\mathcal{O}}^{-1}(0)}\omega_{\mathcal{Y}_{\mathcal{O}}}\big)^{n-d-p}}{(n-d-p)!} = a\circ\varphi_{\mathcal{O}}^{-1}\circ\iota_{\mathrm{Reg}\,\Omega}\,\frac{\big(\iota^*_{\mathrm{Reg}\,\mathcal{J}_{\mathcal{O}}^{-1}(0)}\omega_{\mathcal{Y}_{\mathcal{O}}}\big)^{n-d}}{(n-d)!}, \tag{9.14}$$

which is equivalent to (9.7) over $U_{\mathcal{O}}$. □

In order to fully describe the cohomology of the quotient $\mathrm{Reg}\,M_{\mathrm{red}}$, it would still be necessary to consider more general forms $\varrho\in H^*_G(M)$ than the ones examined in Theorem 9.2. For this, one would need a full asymptotic expansion for the integrals studied in Theorem 8.1, and we intend to tackle this problem in a future paper.

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